MATHEMATICAL

QUESTIONS AND SOLUTIONS,

FROM THE "EDUCATIONAL TIMES,"

WITH MANY ADDITIONAL

PAPERS AND SOLUTIONS

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AND

AN APPENDIX.

EDITED BY

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CONTENTS.

Mathematical Papers, &c.

Questions Solved.

MATHEMATICS

FROM

THE EDUCATIONAL TIMES:

WITH ADDITIONAL PAPERS AND SOLUTIONS.

4118. (By Professor SYLVESTER, F.R.S.) — 1. Given four points in a circle, find the equations in rectangular coordinates to the two circular cubics of which they are foci.

2. Find the equations for determining the foci of the two Cartesian ovals having a given axis and passing through four given points in a circle.

Solution by W. J. C. SHARP, M.A.

I have shown (Vol. xxxv., p. 47) that, if, when the equation to any circular cubic is brought into the form

$$x(x^2+y^2) = ax^2+2bxy+2fy+2gx+c,$$

$$A^4+aA^3-2gA^2+cA = (f-bA)^2 \quad \text{.....................(1)},$$

then the foci are determined by the equations

$$A^2h^2+2Afk-f^2+cA = 0, \quad (k-b)^2+2Ah-A(a+A) = 0 \quad \text{...(2, 3)},$$

$$\therefore \quad A^2(h^2+k^2)+2Ak(f-bA)+2A^3h-2gA^2+2hfA-2f^2+2cA = 0.$$

So that concyclic foci correspond to one root of (1) and lie on two parabolas which have their axes at right angles. If the origin be removed to the intersection of these, equation (1) remains unaltered, while those of the cubic and the parabolas become

$$x(x^2+y^2) = ax^2+2fy+2(g+\tfrac{1}{2}b^2)x+c+2bf \quad \text{...............(4)},$$

$$A^2h^2+2Afk-f^2+2Abf+cA = 0 \text{ and } k^2+2Ah-A(a+A) = 0.$$

Now the four given points determine two parabolas with their axes at right angles. Let the equations to these referred to their axes be

$$x^2+2By+C = 0 \text{ and } y^2+2Dy+E = 0,$$

which may be identified with the two focal parabolas in two ways. Identifying the first with the first, and the second with the second,

$$B = \frac{f}{A}, \quad C = \frac{c+2bf}{A}-\frac{f^2}{A^2}, \quad D = A, \quad E = -A(a+A),$$

and $$A^4+aA^3-2(g+\tfrac{1}{2}b^2)A^2+(c+2bf)A+f^2 = 0,$$

which fully determine the coefficients in (4), and so the circular cubic.

The other may be determined in the same way by identifying the first focal parabola with the second through the points.

If r_1, r_2, r_3, r_4 and r_1', r_2', r_3', r_4' be the distances of the four given points from the two foci of the Cartesian, the equations

$$lr_1+mr_1' = n,\quad lr_2+mr_2' = n,\quad lr_3+mr_3' = n,\quad \text{and}\quad lr_4+mr_4' = n,$$

must hold by the definition of the curve. Consequently

$$\left\| \begin{matrix} r_1, & r_2, & r_3, & r_4 \\ r_1', & r_2', & r_3', & r_4' \\ 1, & 1, & 1, & 1 \end{matrix} \right\| = 0,$$

and these are the conditions that the foci of the Cartesian should lie on a circular cubic having the four given points for concyclic foci. Hence the foci are the points in which the given axis meets the circular cubic, which is determined by the equations in part 1 of the question. As there are two cubics, each giving a set of foci, there will be two Cartesians, which will each have three axial foci (a property of the Cartesian which is thus deducible from the definition), except in the case when the given axis is parallel to the axis of one of the parabolas. In this case, one of circular cubics will only give two, or, when the axis coincides with that of the parabola, only one, focus for the Cartesian, which degenerates into a central conic or a parabola (one of those already found).

7862. (By Professor Haughton, F.R.S.)—A condition of stable equilibrium of heat is produced in a ring, represented by the equation $\dfrac{d^2v}{dx^2} = a^2v$; if the temperatures v_1, v_2, v_3, &c. be taken at equal distances along the ring, show (1) that $v+v_3 = qv_2$, $v_2+v_4 = qv_3$, &c.; and (2) verify the law by means of the following observations:—The temperatures observed were as follows, the distance of the points being 45°:— $v = 66°$, $v_4 =$ unknown; $v_2 = 50\frac{7}{12}°$; $v_5 = 52°$; $v_3 = 44°$, air $= 17\frac{3}{8}°$.

Solution by Âsûtosh Mukhopâdhyây.

The equation of the stability of equilibrium of heat in the ring is

$$\frac{d^2v}{dx^2} = a^2v \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

where a^2 is a constant, depending on l, s, h, k,—l being the perimeter of the section whose area is s, and h, K the coefficients of external and internal conducibility respectively, viz., we have $a^2 = \dfrac{hl}{ks}$.

The solution of (1) is $v = \mathrm{M}e^{-xa}+\mathrm{N}e^{+xa}$, M and N being the two constants of integration. Suppose that the ring is divided into a number of equal parts, and let v_1, v_2, v_3, ... be the temperatures corresponding to the distances x_1, x_2, x_3, ... from the origin. Then, writing $e^{-a} = \alpha$, and

$x_2 - x_1 = \lambda$ = distance between two consecutive points of division, we have

$$v_1 = \mathrm{M}a^{+x_1} + \mathrm{N}a^{-x_1}, \quad v_2 = \mathrm{M}a^{+\lambda}a^{+x_1} + \mathrm{N}a^{+\lambda}a^{-x_1},$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$v_n = \mathrm{M}a^{+(n-1)\lambda}a^{+x_1} + \mathrm{N}a^{-(n-1)\lambda}a^{-x_1},$$

$$\therefore \qquad v_n + v_{n+2} = \mathrm{M}a^{+x_1}a^{+n\lambda}(a^{+\lambda} + a^{-\lambda}) + \mathrm{N}a^{-x_1}a^{-n\lambda}(a^{+\lambda} + a^{-\lambda})$$
$$= v_{n+1}(a^{+\lambda} + a^{-\lambda}).$$

If q be the constant value of $(a^{+\lambda} + a^{-\lambda})$, we have $\frac{v_n + v_{n+2}}{v_{n+1}} = q$, which is the equation required. Writing for n successively 1, 2, 3 ... &c., we get

$$\frac{v_1 + v_3}{v_2} = \frac{v_2 + v_4}{v_3} = \frac{v_3 + v_5}{v_4}.$$

If $v_1 = 66$, $v_2 = 50\frac{7}{12}$, $v_3 = 44$, $v_5 = 52$, this gives the two equations

$$\frac{v_4 + 50\frac{7}{12}}{44} = \frac{66 + 44}{50\frac{7}{12}}, \quad \frac{44 + 52}{v_4} = \frac{66 + 44}{50\frac{7}{12}} \quad \ldots\ldots\ldots\ldots\ldots\ldots(2, 3).$$

From (2) and (3) we have $v_4 = 45\cdot10035945$, $v_4 = 44\cdot1\dot{4}\dot{5}$. The difference between the two values is $\cdot9549$, which is less than unity, and quite within the limits of experimental errors.

6428. (By W. R. Roberts, M.A.)—Prove that the developable formed by the tangent lines of the curve of intersection of

$$\mathrm{U} \equiv ax^2 + by^2 + cz^2 + du^2, \quad \mathrm{V} \equiv a'x^2 + b'y^2 + c'z^2 + d'u^2,$$

can be written
$$x\left\{(da')(bc')(a\mathrm{V} - a'\mathrm{U})\right\}^{\frac{1}{2}}$$
$$+ y\left\{(db')(ca')(b\mathrm{V} - b'\mathrm{U})\right\}^{\frac{1}{2}} + z\left\{(dc')(ab')(c\mathrm{V} - c'\mathrm{U})\right\}^{\frac{1}{2}} = 0.$$

[The above form shows that the sections by the principal planes are double curves, which are easily seen to be Lemniscates, having the vertices of the tetrahedron of reference as nodes.]

Solution by W. J. C. Sharp, M.A.

If $(\xi, \eta, \zeta, \theta)$ be any point on the tangent to the curve of intersection at the point (x, y, z, w), $(\xi, \eta, \zeta, \theta)$ is a point on the developable, and its coordinates will satisfy the eliminant of the equations

$$ax^2 + by^2 + cz^2 + dw^2 = 0, \quad a'x^2 + b'y^2 + c'z^2 + d'w^2 = 0 \ldots\ldots\ldots(1, 2),$$
$$ax\xi + by\eta + cz\zeta + dw\theta = 0, \quad a'x\xi + b'y\eta + c'z\zeta + d'w\theta = 0 \ldots\ldots\ldots(3, 4).$$

(3) and (4) may be replaced by $(da')\,x\xi + (db')\,y\eta + (dc')\,z\zeta = 0$ ....(5), and the result of eliminating $xy\xi\eta$, and $zw\zeta\theta$ between

$$a^2x^2\xi^2 + b^2y^2\eta^2 - c^2z^2\zeta^2 - d^2w^2\theta^2 + 2abxy\xi\eta - 2cdzw\zeta\theta = 0,$$
$$a'^2x^2\xi^2 + b'^2y^2\eta^2 - c'^2z^2\zeta^2 - d'^2w^2\theta^2 + 2a'b'xy\xi\eta - 2c'd'zw\zeta\theta = 0,$$
$$aa'x^2\xi^2 + bb'y^2\eta^2 - cc'z^2\zeta^2 - dd'w^2\theta^2 + (ab' + a'b)\,xy\xi\eta - (cd' + c'd)\,zw\zeta\theta = 0,$$

which, when expressed as a determinant, reduces to

$$(ab')(ac')(ad')\,x^2\xi^2 + (bc')(bd')(ba')\,y^2\eta^2$$
$$+ (cd')(cc')(cb')\,z^2\zeta^2 + (da')(db')(dc')\,w^2\theta^2 = 0,$$

from which, and the equations (1) and (2),

$$x^2 : y^2 : z^2 : w^2 :: (bc')(cd')(db') \left\{ (ba')\,\eta^2 + (ca')\,\zeta^2 + (da')\,\theta^2 \right\}$$
$$: (cd')(da')(ac') \left\{ (ab')\,\xi^2 + (cb')\,\zeta^2 + (db')\,\theta' \right\} : \&c.,$$

or as $\quad (bc')(cd')(db') \left\{ a'\mathrm{U} - a\mathrm{V} \right\} : (cd')(da')(ac') \left\{ b'\mathrm{U} - b\mathrm{V} \right\} : \&c.,$

and therefore, by substitution in (5),

$$(da')\,\xi \left\{ (bc')(cd')(db')(a'\mathrm{U} - a\mathrm{V}) \right\}^{\frac{1}{2}} + (db')\,\eta \left\{ (cd')(da')(ac')(b'\mathrm{U} - b\mathrm{V}) \right\}^{\frac{1}{2}}$$
$$+ (dc')\,\zeta \left\{ (da')(ab')(bd')(c'\mathrm{U} - c\mathrm{V}) \right\}^{\frac{1}{2}} = 0,$$

or
$$\xi \left\{ (da')(bc')(a\mathrm{V} - a'\mathrm{U}) \right\}^{\frac{1}{2}} + \eta \left\{ (db')(ca')(b\mathrm{V} - b'\mathrm{U}) \right\}^{\frac{1}{2}}$$
$$+ \zeta \left\{ (dc')(ab')(c\mathrm{V} - c'\mathrm{U}) \right\}^{\frac{1}{2}} = 0.$$

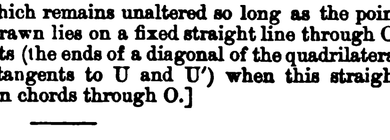
The sections by the principal planes are obtained as in the question.

7842. (By Professor Wolstenholme, M.A., Sc.D.)—Two confocal conics, U, U′ have a common chord OO′ (perpendicular to the focal axis), and from a point of this chord are drawn tangents to U, meeting the tangent to U at O in P, Q, and tangents to U′ meeting the tangent to U′ at O in P′, Q′: prove that (1) PP′, PQ′, QP′, QQ′ pass each through one of the foci; (2) also, if tangents from any point to U meet the tangent to U at O in p, q, and tangents from the same point to U′ meet the tangent to U′ at O in p', q', the four straight lines pp', pq', qp', qq' all touch a conic confocal with U, U′; which degenerates into two points when the point from which the tangents are drawn lies on one of the common chords through O, and which remains the same so long as the point from which the tangents are drawn lies on a fixed straight line through O.

[*Generalized by projection*, the theorem is as follows :—

Two conics U, U′ intersect in O, and tangents drawn from any point to U meet the tangent to U at O in P, Q, and tangents drawn from the same point to U′ meet the tangent to U′ at O in P′, Q′; the four straight lines PP′, PQ′, QP′, QQ′ all touch a conic which touches the four common tangents to U and U′; and which remains unaltered so long as the point from which the tangents are drawn lies on a fixed straight line through O, but degenerates into two points (the ends of a diagonal of the quadrilateral formed by the four common tangents to U and U′) when this straight line is one of the three common chords through O.]

Solution by Arthur Hill Curtis, LL.D., D.Sc.



Let AEB, A′E′B′ be two circles, whose diameters are D, D′, whose centres are C, C′, and a pair of whose common tangents, touching them at E, E_1, E′, E'_1, meet at F_1; and let these circles, for purpose of reference, be denoted

by (C), (C′); let ABA′B′ be any line, denoted by R, cutting them, and p, p', the perpendiculars from EE′ on this line, let AE, B′E′ meet in L_1, $\therefore AL_1 . L_1E : B'L_1 . L_1E' = \sin E' \sin B' : \sin E \sin A = \sin A' \sin B' : \sin A \sin B$, but $\sin A \sin B = AE . BE \div D^2 = pD \div D^2 = p \div D$, while $\sin A' \sin B' = p' \div D'$,

therefore $\quad AL_1 . L_1E : B'L_1 . L_1E' = pD' : p'D = EH . D' : E'H . D.$

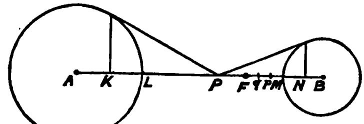
Therefore, so long as R passes through the fixed point H, the ratio $BL_1 \times L_1E : A'L_1 \times L_1E'$ is constant $= k$, suppose, and therefore L_1 lies on a circle having with the circles (C), (C′) a common radical axis, and if L_2, L_3, L_4 be the intersections of the lines BE, B′E′; AE, A′E′; BE, A′E′, the same is true, *k being unaltered;* therefore the four points, L_1, L_2, L_3, L_4 lie on a circle having with the circles (C), (C′) a common radical axis...(*a*),

When H coincides with F_1, $k = 1$, and each of the points L_1, L_2, L_3, L_4, either lies on the radical axis of the circles (C), (C′), or goes to infinity...(*b*).

If we reciprocate the results (*a*) and (*b*), taking for origin one of the limiting points of the system of circles, to which (C) and (C′) belong, we obtain from (*a*) the theorem (2), and from (*b*) the theorem (1), for parallel lines reciprocate into points the line joining which passes through the origin, the second limiting point into the centre of the confocal system into which the system of circles reciprocates, and therefore the radical axis, as it bisects perpendicularly the line joining these limiting points, into the second focus of the confocal system. If we project the system of confocal conics into a system of conics inscribed in the same quadrilateral, we obtain the final theorem of the question.

7868. (By the Editor.) — In the line joining the centres of two spheres, find geometrically a point such that the sum of the surfaces of the spheres visible therefrom shall be a maximum.

Solutions by (1) Rev. T. C. Simmons, M.A.; (2) D. Biddle.

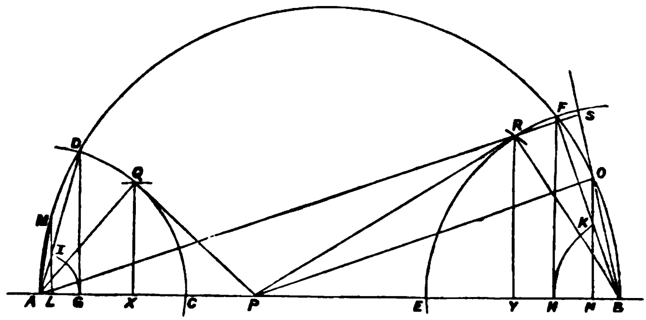


1. Let A, B be the centres of the spheres; R, r their radii; P any point in AB; and K, N the points where AB is met by the two polar planes of P; then we require a maximum for

$$R . KL + r . MN,$$

i.e., a minimum for $R . AK + r . BN$, or for $\dfrac{R^3}{AP} + \dfrac{r^3}{BP}$. Take $R : r_1$ in the triplicate ratio of R to r, and let r_2 be the mean proportional between R and r_1; then the required point F will be found by dividing AB so that $AF : FB = R : r_2$. For we have $AF^2 : FB^2 = R : r_1 = R^3 : r^3$, so that the minimum is required for $\dfrac{AF^2}{AP} + \dfrac{BF^2}{BP}$. If P do not coincide with F, take $AP . Ap = AF^2$, and $BP . Bq = BF^2$; then, from the fact that the arithmetic mean between two lines exceeds the geometric mean, we have

$Fp > FP > Fq$, showing that $Ap + Bq$ is greater than AB, and consequently that, when P does not coincide with F, the value of $\frac{AF^2}{AP} + \frac{BF^2}{BP}$ is always greater than when it does.



2. *Otherwise.*—Let A, B be the respective centres, and CQD, ERF arcs of great circles in the same plane in which AB lies; then, since $2\pi rh =$ the superficial area of any segment of a sphere,

$$2\pi \,.\, AQ \,.\, CX \left[= 2\pi\left(AQ^2 - \frac{AQ^3}{AP}\right)\right],$$

and $$2\pi \,.\, BR \,.\, EY \left[= 2\pi\left(BR^2 - \frac{BR^3}{BP}\right)\right]$$

are the respective surfaces visible from P. If a and b represent the respective radii of the spheres, $\frac{a^3}{AP^2}$ and $\frac{b^3}{BP^2}$ are the differential coefficients for these surfaces; and when they are equal, and $a^3 : b^3 = AP^2 : BP^2$, or $a^{\frac{3}{2}} : b^{\frac{3}{2}} = AP : BP$, the point P is the point required. On AB draw a semi-circle cutting the surface of the spheres in D and F respectively. Join AD and BF. Draw DG, FH perpendicular to AB, and from A and B as centres describe arcs GI, HK cutting AD, BF in I and K respectively. Through I and K draw LM, NO at right angles with AB so as to cut the semi-circle in M and O. Join AM, BO, and produce BO to S, making OS = AM. Join SA, and draw OP parallel to SA. Then

$$AM\,(= SO) : AP = BO : BP.$$

But, AB being taken as unity, $BO = BN^{\frac{1}{2}}$, and $BN : BK = BH : BF$, therefore $BN = BF^3$, and $BO = BF^{\frac{3}{2}}$. Similarly $AM = AD^{\frac{3}{2}}$. Therefore $AP : BP = AD^{\frac{3}{2}} : BF^{\frac{3}{2}} = a^{\frac{3}{2}} : b^{\frac{3}{2}}$, and P is the point required.

7762. (By Rev. T. C. Simmons, M.A.)—Assuming the wear and tear of a gold coin in circulation during a given time to be proportional to the area of its surface, and considering a sovereign as a plane-faced circular disc whose diameter is approximately 15 times its thickness, find what this

multiple ought to be in the case of the half-sovereign to make the percentage of loss (1) the same, (2) 1·8 times as much, as for the sovereign.

Solution by the PROPOSER; W. G. LAX, B.A.; *and others.*

Let t be thickness of sovereign, τ and $x\tau$ thickness and diameter of half-sovereign; then the areas are respectively $15\pi t^2 + \frac{225}{2}\pi t^2$ and $x\pi\tau^2 + \frac{1}{2}x^2\pi\tau^2$, the ratio of which is $\frac{255t^2}{(x^2+2x)\tau^2}$. Now, in the first case, this ratio ought to $= 2$, whence, substituting $\frac{t^2}{\tau^2} = \left(\frac{2x^2}{225}\right)^{\frac{2}{3}}$, we obtain $\frac{2x^2+4x}{255} = \frac{x^{\frac{4}{3}}}{(112\cdot5)^{\frac{2}{3}}}$, which, after dividing by x, reduces to $x - 5\cdot471x^{\frac{1}{3}} + 2 = 0$,

giving $\quad x^{\frac{1}{3}} = 2\cdot13$, or *the required multiple* $= 9\cdot6$.

In the second case, the above ratio must $= \frac{10}{9}$, whence

$$\frac{x^2+2x}{255} = \frac{9}{10}\,\frac{x^{\frac{4}{3}}}{(112\cdot5)^{\frac{2}{3}}}, \text{ reducing to } x - 9\cdot858x^{\frac{1}{3}} + 2 = 0,$$

whence $\quad x^{\frac{1}{3}} = 3\cdot033$, or *the required multiple* $= 27\cdot9$.

NOTE (WITH REFERENCE TO QUESTION 7762) ON THE RELATIVE WEAR AND TEAR OF SOVEREIGNS AND HALF-SOVEREIGNS.

By Rev. C T. SIMMONS, M.A.

This question was suggested by a letter in the *Times* of May 1, in which it was pointed out that the percentage of loss due to wear and tear in the half-sovereign as compared with the sovereign ought, from purely geometrical reasons, to be as $\sqrt[3]{2} : 1$; and that, as the actual relative wear and tear had been stated by Mr. CHILDERS to be about 2 : 1, the difference was probably to be accounted for by the large employment of sovereigns in storage for bullion and other purposes. The following considerations will make it appear that the first of these statements was erroneous, and the second, to say the least, questionable.

1. The coins are not, as is generally assumed, similar solids. On placing a sufficient number of them upon each other, it will be found that the ratio of diameter to thickness is, in new sovereigns very nearly 15, and in new half-sovereigns very nearly 18. These data are sufficient for a comparison of the superficial areas. Taking t to denote the thickness, r the radius of a sovereign, τ, ρ corresponding quantities for the half-sovereign, the areas will be in the ratio $r(r+t) : \rho(\rho+\tau)$, or $15.16.t^2 : 18.19.\tau^2$. Now by comparing the volumes we obtain $r^2t = 2\rho^2\tau$, or $225t^3 = 648\tau^3$, whence

$$\frac{\text{area of half-sovereign}}{\text{area of sovereign}} = \frac{18.19.\tau^2}{15.16.t^2} = \frac{57}{40}\left(\frac{225}{648}\right)^{\frac{2}{3}} = \cdot7039\ldots;$$

so that, assuming the wear and tear to be proportional to the area of the surface, and that the roughnesses due to the embossing may be left out of account, as probably affecting both coins to the same extent, we see that the wear and tear of the half-sovereign ought, with reference to its weight, to be twice ·7039 or 1·4 times as much as that of the sovereign.

2. Now, what is the actual proportion of wear and tear? Mr. Childers I believe, stated it to be about 2 : 1. He possibly obtained the statement from a Report addressed to the Chancellor of the Exchequer by the Master of the Mint in 1869 (reprinted in the *Journal of the Institute of Bankers* for April, 1884), in which the legal lifetime of the average sovereign is given as 18 years, and that of the average half-sovereign 10 years. In another part of the same Report the relative wear and tear is given as 1·024 to ·43, or as nearly 2·4 to 1, a discrepancy which will be alluded to further on. We will for the present take it to be as 1·8 to 1; and this compared with the above-determined ratio of 1·4 to 1 would lead us to conclude that the actual wear and tear of half-sovereigns is about $1\frac{1}{3}$ times as much as, from purely geometrical considerations, it ought to be.

3. How then, is this to be accounted for? We will make the same comparison in the case of shillings and sixpences. The ratio of diameter to thickness in new shillings is approximately 14, and in new sixpences approximately 15. Proceeding as above, this gives the areas in the ratio of ·6567... : 1. The actual wear and tear (see Prof. Jevons' *Treatise on Money*, in *International Science Series*, p. 158) has been found to be as 1·875 : 1. Comparing 1·875 with 1·31, we conclude, then, that the wear and tear of the sixpence, considered with reference to the shilling, is rather more than 1·4 times what it ought to be. So that whatever causes are at work to increase the relative wear and tear of the half-sovereign (or, which is the same thing, to diminish the relative wear and tear of the sovereign) are to be found acting in a still greater degree in the case of the sixpence as compared with the shilling. There is consequently no need to resort to the bullion hypothesis. Or if it be said from *à priori* considerations that this must exert an appreciable influence, then we must conclude that the rapidity of circulation (or whatever equivalent phrase we adopt) of sixpences as compared with shillings is much greater than that of half-sovereigns as compared with sovereigns. Whether this is consonant with experience, everyone must judge for himself.

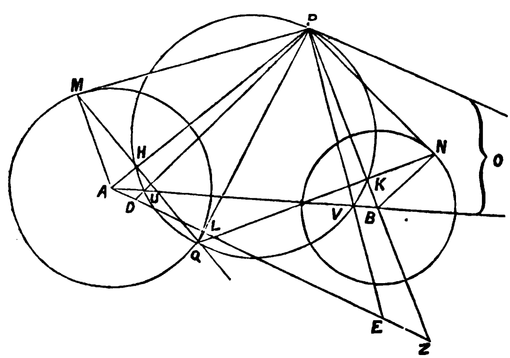
4. It only remains to notice the above alluded-to discrepancy between 1·8 and 2·4 as representing the relative wear and tear of the half-sovereign. In Prof. Jevons' *Treatise on Money*, already quoted, it is given as still greater, namely, $\frac{112}{35}$ or more than 3 (p. 158). I have been unable to obtain any explanation of these inconsistent statements, and can only, wrongly or rightly, suggest the following. It may be that, in investigations concerning the legal lifetime of coins, only those are considered which are still legally current. For instance, a large number of comparatively young half-sovereigns might be taken, and it might be found that their average annual loss was ·0387 grains, which would give the average legal lifetime 10 years. But now what would happen if older half-sovereigns were taken? The influence of what is known in Political Economy as Gresham's Law would have come into play. The least-worn coins would gradually have gravitated back to the Bank of England; the lightest ones would, for obvious reasons, not have been taken there, but would have remained in circulation. So that, when old half-sovereigns are included, the average annual wear and tear must almost certainly be found to be greater than when those of comparatively recent issue are considered alone.

[Perhaps some of our correspondents may be able to throw light on this question, and, if the above explanation is inadequate or wrong, suggest another.]

7760. (By MORGAN JENKINS, M.A.)—Give a geometrical construction for proving independently (and not as a deduction from a special case) that the locus of a point which moves so that the tangents drawn from it to two given circles are in a constant ratio, is a coaxal circle.

Solution by the PROPOSER; J. MCDOWELL, M.A.; *and others.*

Let A, B be the centres of the two circles; PM, PN the two tangents; MH, NK, drawn perpendicular to PA, PB respectively, meet in Q. Describe



a circle on PQ as diameter passing through H and K, and meeting AB in U and V. Draw PO a tangent to this circle at P and meeting AB produced in O. Draw a straight line ADL parallel to PO, and meeting PU, PQ, PV, PB (produced if necessary) in D, L, E, and Z.

Then, PQ being a diameter of the circle HQK (because PHQ is a right angle), and perpendicular to AZ, the rectangles PH . PA, PU . PD, PQ . PL, PV . PE, PK . PZ are equal to one another, and to the square on PM. And $PK \cdot PB = PN^2$; hence $PK \cdot PZ : PK \cdot PB = k^2 : 1$, if k be the given ratio. And $AO : BO = PZ : PB = k^2 : 1$, hence O is a fixed point. Also $AU \cdot AV = AH \cdot AP = AM^2$; and $BV \cdot BU = BN^2$, therefore U and V are fixed points, viz., the limiting points of the system of coaxal circles determined by the given circles. And the square on OP = rectangle OU . OV, which is fixed in magnitude; therefore the locus of P is a circle, with centre O, and coaxal with the system of circles having U and V for limiting points, that is, the system determined by the given circles. Or, again, $PU : PD = OU : AO$ and $PE : PV = AO : OV$, therefore $PU \cdot PE : PV \cdot PD = OU : OV$; but $PU \cdot PD = PV \cdot PE$, therefore $PU^2 : PV^2 = OU : OV$, a constant ratio, whence the same result follows. Also

$$PM^2 : PV^2 = PV \cdot PE : PV^2 = PE : PV = AO : VO = \text{a constant ratio},$$

and $$PM^2 : PU^2 = PU \cdot PD : PU^2 = PD : PU = AO : UO,$$

a constant ratio, with similar results for PU and PN, and for PV and PN. Hence for either of the two given circles we may substitute either of the two limiting points U and V, or any other circle of the coaxal

system, and, by properly altering the value of the given ratio, obtain the same circle as before.

[Another solution to this question is indicated in McDOWELL's *Exercises on Euclid and in Modern Geometry*, No. 270, foot of p. 237, new edition.]

7843. (By Professor HUDSON, M.A.)—A particle moves in an orbit about a luminous centre of force, and casts a shadow on the inverse of the orbit with respect to the luminous point; the shadow moves as if in an orbit about the luminous centre: show that the orbit is a circle, whose centre coincides with the centre of force.

Solution by Dr. CURTIS; Professor NASH, M.A.; *and others.*

The following equations must hold good:—

$$r_1^2 d\theta = h_1 dt, \quad r_2^2 d\theta = h_2 dt, \quad r_1 r_2 = k^2,$$

therefore $\dfrac{h_1}{h_2} = \dfrac{r_1^2}{r_2^2} = \dfrac{r_1^4}{k^4}$, therefore r_1 is constant, therefore &c.

7726. (By J. W. RUSSELL, M.A.)—Prove *geometrically* that, at the intersection of two confocal conics, the centre of curvature of either is the pole with respect to the other of the tangent to the former at the intersection.

Solution (*communicated by* Dr. CURTIS) *by the late* Prof. TOWNSEND, F.R.S.

Let C, D be two consecutive points on a conic A, then, as the normal at the point C is the locus of the pole of the tangent CD with regard to all confocal conics, the pole of this tangent taken with regard to any such conic B is the intersection of this normal with the B polar of D; if now the conic be supposed to be the confocal through D, this polar becomes the tangent to B at D, and therefore a normal to the conic A; the B pole of CD is then the intersection of two consecutive normals to the conic A, viz., the centre of curvature.

7880. (By SARAH MARKS.)—120 men are to be formed at random into a solid rectangle of 12 men by 10, all sides being equally likely to be in front; show that the chance that an assigned man is in the front is $\frac{11}{120}$.

Solution by D. Biddle; Rev. T. C. Simmons, M.A.; *and others.*

There are 40 positions having a chance of being in front, 36 having $\frac{1}{4}$ chance, and 4 (the corner positions) having $\frac{1}{2}$ chance. Hence we have

$$P = \frac{(36 \times \frac{1}{4}) + (4 \times \frac{1}{2})}{120} = \frac{11}{120}.$$

7717. (By R. Tucker, M.A.)—The circles about AEF, BFD, CDE cointersect in O, and those about AE′F′, BF′D′, CD′E′ cointersect in O′: also the triangles formed by joining the centres of the two sets of circles are similar to the primitive triangle ABC, and equal to one another. Find the ratio of similitude in terms of Brocard's angle.

Solution by B. Hanumanta Rau, M.A.; E. Rutter; *and others.*

If α, β, γ be the distances of P from the sides of the primitive triangle,

we have $$\frac{\alpha}{a} = \frac{\beta}{b} = \frac{\gamma}{c} = \frac{2\Delta}{a^2+b^2+c^2},$$

$$CD = a\,(a^2+b^2)\,/\,a^2+b^2+c^2 \text{ and } CE = a^2b\,/\,a^2+b^2+c^2.$$

The sides BC, CA, AB subtend angles A + B, B + C, C + A at the first Brocard-point O [see solution of Quest. 7758, Vol. 32, p. 113];

$$\therefore \frac{a\,.\,OA}{\sin B} = \frac{b\,.\,OB}{\sin C} = \frac{c\,.\,OC}{\sin A} = \frac{abc}{OA\sin C + OB\,.\,\sin A + OC\sin B},$$

$$\therefore CO = \frac{a^2b}{[b^2c^2+c^2a^2+a^2b^2]^{\frac{1}{2}}}, \quad \therefore \frac{CO}{2CE} = \frac{a^2+b^2+c^2}{2\,[b^2c^2+c^2a^2+a^2b^2]^{\frac{1}{2}}}$$
$$= \cos\theta = \cos\angle OCA,$$

therefore $\quad EO = EC$ or $\angle EOC = \angle ECO = \angle OBC = \angle EDC$.

Therefore the circle about CDE passes through O, similarly DO = DB and FO = FA, or the circles about BDF and AFE pass through the same B-point O.

The circles about AE′F′, BF′D′, and CD′E′ cointersect in the second B-point O′.

Again, $$\cot\theta = \cot A + \cot B + \cot C = \frac{a^2+b^2+c^2}{4\Delta},$$

therefore $$\frac{\cos\theta}{a^2+b^2+c^2} = \frac{\sin\theta}{4\Delta} = \frac{1}{2\,[b^2c^2+c^2a^2+a^2b^2]^{\frac{1}{2}}},$$

$$DE^2 = CD^2 + CE^2 - 2CD\,.\,CE\cos C = \tfrac{1}{4}a^2\sec^2\theta,$$

Let Q, Q_1, Q_2, Q_3 be the centres of the circles about DEF, AEF, BFD, and CDE; then $$QQ_3 = \frac{DE}{2}(\cot A + \cot C) = \frac{ab^2}{8\Delta\cos\theta},$$

$$QQ_2 = \frac{DF}{2}(\cot B + \cot C) = \frac{a^2c}{8\Delta\cos\theta},$$

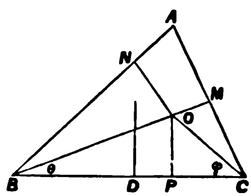
therefore $$Q_2Q_3 = \frac{a}{8\Delta\cos\theta}\left[b^4 + a^2c^2 + 2ab^2c\cos B\right]^{\frac{1}{2}}$$
$$= \frac{a}{8\Delta\cos\theta}\cdot\frac{2\Delta}{\sin\theta} = \frac{a}{2\sin 2\theta}.$$

Therefore the sides of the triangle $Q_1Q_2Q_3$ are proportional to the sides a, b, c, and therefore the triangle $Q_1Q_2Q_3$ is similar to the triangle ABC and the ratio of similitude is $\frac{1}{2}$ cosec 2θ.

[If DEF be *any* inscribed triangle, the circles AEF, &c. will pass through the same point O (*Diary* for 1859, Question 1934), therefore $\angle\,\mathrm{EOF} = \pi - \mathrm{A} = \mathrm{B} + \mathrm{C}$, and therefore (as in above proof), since DEF, ABC are similar, O is the first B-point of DEF; and similar results hold for the point O′. In the same *Diary* question, it is shown geometrically that $Q_1Q_2Q_3$ is, for *any* inscribed triangle, similar to ABC.]

7614. (By R. Tucker, M.A.)—The base and vertical angle of a triangle being given, prove that the locus of the *point de Grebe* (*i.e.*, "Symmedian" point), and therefore also of the "Triplicate-centre," is an ellipse, which, in the former case, if $\lambda^{-1} = 4 - \cos^2 A$, can be put into the form $\frac{x^2}{a^2\lambda} + \frac{y^2\,\mathrm{cosec}^2 A}{a^2\lambda^2} = 1.$

Solution by B. Hanumanta Rau, M.A.



Let O be the *point de Grebe*; then
$$\mathrm{OP} : \mathrm{OM} : \mathrm{ON} = a : b : c$$
$$= \sin A : \sin B : \sin C;$$
and, taking D as the origin of coordinates,
$$\tfrac{1}{2}a + x = y\cot\theta, \quad\text{and}\quad \tfrac{1}{2}a - x = y\cot\phi,$$
$$\frac{\sin(B-\theta)}{\sin\theta} = \frac{\mathrm{ON}}{\mathrm{OP}} = \frac{\sin C}{\sin A},$$
$$\therefore \cot\theta - \cot B = \frac{\sin C}{\sin A\sin B} = \cot A + \cot B.$$

Hence we have $$\cot\theta = \cot A + 2\cot B = \frac{a+2x}{2y};$$

similarly, $$\cot\phi = \cot A + 2\cot C = \frac{a-2x}{2y};$$

but $$4\cot A(\cot B + \cot C) + 4\cot B\cot C \equiv 4;$$

therefore $$4\cot A\left(\frac{a}{2y} - \cot A\right) + \left(\frac{a}{2y} - \cot A\right)^2 - \frac{x^2}{y^2} = 4,$$

or $$x^2 + (4\,\mathrm{cosec}^2 A - \cot^2 A)\,y^2 - a\cot A\,y = \tfrac{1}{4}A^2,$$

$$\therefore\ \lambda^{-1}x^2 + \lambda^{-2}\,\mathrm{cosec}^2 A\,(y - \tfrac{1}{2}a\lambda\cos A\sin A)^2 = \tfrac{1}{4}a^2\lambda^{-1} + \tfrac{1}{4}a^2\cos^2 A = a^2,$$

or $$\frac{x^2}{a^2\lambda} + \frac{y^2 \operatorname{cosec}^2 A}{a^2\lambda^2} = 1.$$

The centre of the T. R.-circle bisects the line joining the fixed circumcentre with the *point de Grebe*. The locus of the T. R. centre is therefore an ellipse of half the dimensions.

7463. (By W. J. C. SHARP, M.A.)—If S_r denote the sum of the r^{th} powers of the roots of $ax^n - p_1x^{n-1} + p_2x^{n-2} - p_3x^{n-3} + \&c. = 0$, prove that

$$S_r = \frac{(-1)^{r-1}}{(r-1)!}\left(p_1\frac{d}{da} + 2p_2\frac{d}{dp_1} + \&c.\right)^{r-1}\left(\frac{p_1}{a}\right),$$

and $$S_{-r} = \frac{(-1)^{r-1}}{(r-1)!}\left(na\frac{d}{dp_1} + (n-1)p_1\frac{d}{dp_2} + \&c.\right)^{r-1}\left(\frac{p_n}{p_{n-1}}\right).$$

Solution by G. B. MATHEWS, B.A.; J. O'REGAN; *and others.*

$$S_{-1} = \frac{1}{\alpha} + \frac{1}{\beta} + \dots = \frac{p_{n-1}}{p_n};$$ hence, writing $x+\lambda$ for x,

$$S'_{-1} = \frac{1}{\alpha+\lambda} + \frac{1}{\beta+\lambda} + \dots = \frac{1}{\alpha}\left(1 - \frac{\lambda}{\alpha} + \dots + (-)^{r-1}\frac{\lambda^{r-1}}{\alpha^{r-1}}\right) + \frac{1}{\beta}(\dots) + \dots;$$

therefore S_{-r} = coefficient of $(-)^{r-1}\lambda^{r-1}$ in S'_{-1}. Now, if $\phi(a, p_1, p_2 \dots p_n)$ be any rational function of $a, p_1, p_2 \dots p_n$, then, when $x+\lambda$ is written for x, $p_1, p_2, \dots$ become $p_1 - na\lambda$, $p_2 - (n-1)p_1\lambda + \frac{1}{2}n(n-1)a\lambda^2$, ...
Thus ϕ becomes

$$\phi - \lambda\left(na\frac{d}{dp_1} + (n-1)p_1\frac{d}{dp_2} + \dots\right)\phi + \lambda^2(\dots) = \phi - \delta\phi\lambda + \dots \text{ say}$$
$$= \phi - \lambda\delta\phi + \frac{\lambda^2}{2!}\delta^2\phi \dots + (-)^{r-1}\frac{\lambda^{r-1}}{(r-1)!}\delta^{r-1}\phi + \dots$$

Hence $$S_{-r} = \frac{(-)^{r-1}}{(r-1)!}\delta^{r-1}S_{-1} = \frac{(-)^{r-1}}{(r-1)!}\delta^{r-1}\left(\frac{p_{n-1}}{p_n}\right).$$

Moreover, S_r is the S_{-r} of the reciprocal equation

$$p_n x^n - p_{n-1}x^{n-1} + \dots + (-)^n a = 0,$$

or $$S_r = \frac{(-)^{r-1}}{(r-1)!}\left\{np_n\frac{d}{dp_{n-1}} + (n-1)p_{n-1}\frac{d}{dp_{n-2}} + \dots\right\}^{r-1}\left(\frac{p_1}{a}\right)$$
$$= \frac{(-)^{r-1}}{(r-1)!}\left\{p_1\frac{d}{da} + 2p_2\frac{d}{dp_1} + \dots\right\}^{r-1}\left(\frac{p_1}{a}\right),$$

writing the operation in $\{\ \}$ in the reverse order.

7435. (By Satish Chandra Basu.)—Find the general value of x from
$$a+b+c = a^2+b^2+c^2 = a^3+b^3+c^3 = a^{2x}+b^{2x}+c^{2x} = 0.$$

Solution by G. B. Mathews, B.A.; Professor Matz, M.A.; *and others.*

$$0 = (a+b+c)^2-(a^2+b^2+c^2) = 2(bc+ca+ab),$$
$$0 = a^3+b^3+c^3 = (a+b+c)(a^2+b^2+c^2-bc-ca-ab)+3abc.$$

Therefore $\qquad bc+ca+ab = 0$ and $abc = 0$,
therefore a, b, c are the roots of $x^3 = 0$, therefore $a = b = c = 0$, and therefore $a^{2x}+b^{2x}+c^{2x} = 0$, for all values of x, of which the real part is positive.

3269. (By the Editor.)—Prove that the chord which joins the points $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$ on the conic $l\alpha^{\frac{1}{2}}+m\beta^{\frac{1}{2}}+n\gamma^{\frac{1}{2}} = 0$ is parallel to
$$\frac{l\alpha}{\alpha_1^{\frac{1}{2}}+\alpha_2^{\frac{1}{2}}}+\frac{m\beta}{\beta_1^{\frac{1}{2}}+\beta_2^{\frac{1}{2}}}+\frac{n\gamma}{\gamma_1^{\frac{1}{2}}+\gamma_2^{\frac{1}{2}}} = 0.$$

Note by the Editor.

1. Mr. Simmons remarks that the solution of this question, given on p. 47 of Vol. xl., is clearly wrong. For by the same method it would follow that all lines included under the form
$$\frac{l\alpha}{(1-p)\,\alpha_1^{\frac{1}{2}}+(1+p)\,\alpha_2^{\frac{1}{2}}}+\frac{m\beta}{(1-q)\,\beta_1^{\frac{1}{2}}+(1+q)\,\beta_2^{\frac{1}{2}}}+\frac{n\gamma}{(1-r)\,\gamma_1^{\frac{1}{2}}+(1+r)\,\gamma_2^{\frac{1}{2}}} = 0,$$
are parallel; which cannot be the case, as by properly choosing p, q, r the above equation may be made to represent any straight line whatever.

The fallacy in the solution is not easily seen at first, and is only apparent on a close analysis of what is meant by moving a line parallel to itself. When this operation is performed on
$$l\alpha\,[(\beta_1\gamma_2)^{\frac{1}{2}}+(\beta_2\gamma_1)^{\frac{1}{2}}]+m\beta\,[(\gamma_1\alpha_2)^{\frac{1}{2}}+(\alpha_1\gamma_2)^{\frac{1}{2}}]+n\gamma\,[(\alpha_1\beta_2)^{\frac{1}{2}}+(\alpha_2\beta_1)^{\frac{1}{2}}] = 0,$$
what we tacitly do is to take two new points $(\alpha_1'\beta_1'\gamma_1')$, $(\alpha_2'\beta_2'\gamma_2')$ on the curve, so chosen as to ensure the parallelism of the new chord to the old; it follows, *by a repetition of what has gone before*, that the equation of the new line is also similar in form to that of the old, and that the similarity still holds when the two new points coalesce, thus giving the equation of the parallel tangent. But now let us move the line
$$\frac{l\alpha}{\alpha_1^{\frac{1}{2}}+\alpha_2^{\frac{1}{2}}}+\frac{m\beta}{\beta_1^{\frac{1}{2}}+\beta_2^{\frac{1}{2}}}+\frac{n\gamma}{\gamma_1^{\frac{1}{2}}+\gamma_2^{\frac{1}{2}}} = 0.$$
If, as before, the new points are so chosen as to ensure the parallelism, what is there to infer the similarity of the equations? Nothing; the mode of inference in the former case (printed in italics) not being available here.

If, on the other hand (and this is what the solution seems tacitly to imply), the new points are so chosen as to ensure the similarity of the

equations, there is nothing to warrant the inference of parallelism. So that the place of the coalescence of the two new points is not the same for the equation of the chord as it is for the equation given in the question, and the solution collapses.

2. Mr. SHARP has also pointed out the inaccuracy of the above-cited solution, but states the theorem is quite correct, and sends the following proof of it :—

If $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$ lie on the conic $l\alpha^{\frac12}+m\beta^{\frac12}+n\gamma^{\frac12} = 0$,

$$l : m : n = (\beta_1\gamma_2)^{\frac12}-(\beta_2\gamma_1)^{\frac12} : (\gamma_1\alpha_2)^{\frac12}-(\gamma_2\alpha_1)^{\frac12} : (\alpha_1\beta_2)^{\frac12}-(\alpha_2\beta_1)^{\frac12},$$

and the equation $\dfrac{l\alpha}{\alpha_1^{\frac12}+\alpha_2^{\frac12}}+\dfrac{m\beta}{\beta_1^{\frac12}+\beta_2^{\frac12}}+\dfrac{n\gamma}{\gamma_1^{\frac12}+\gamma_2^{\frac12}} = 0$, is equivalent to

$$\frac{\alpha\,[(\beta_1\gamma_2)^{\frac12}-(\beta_2\gamma_1)^{\frac12}][\alpha_1^{\frac12}-\alpha_2^{\frac12}]}{\alpha_1-\alpha_2}+\frac{\beta\,[(\gamma_1\alpha_2)^{\frac12}-(\gamma_2\alpha_1)^{\frac12}][\beta_1^{\frac12}-\beta_2^{\frac12}]}{\beta_1-\beta_2}$$
$$+\frac{\gamma\,[(\alpha_1\beta_2)^{\frac12}-(\alpha_2\beta_1)^{\frac12}][\gamma_1^{\frac12}-\gamma_2^{\frac12}]}{\gamma_1-\gamma_2} = 0 \qquad\ldots\ldots\ldots\ldots(1).$$

If (1) be parallel to the chord

$$\alpha\,(\beta_1\gamma_2-\beta_2\gamma_1)+\beta\,(\gamma_1\alpha_2-\gamma_2\alpha_1)+\gamma\,(\alpha_1\beta_2-\alpha_2\beta_1) = 0 \qquad\ldots\ldots\ldots\ldots(2),$$

(1) and (2) must meet at infinity $A\alpha+B\beta+C\gamma = 0$, the condition for which is

$$\begin{vmatrix} (\beta_1\gamma_2-\beta_2\gamma_1)(\alpha_1-\alpha_2), & [(\beta_1\gamma_2)^{\frac12}-(\beta_2\gamma_1)^{\frac12}]\,[\alpha_1^{\frac12}-\alpha_2^{\frac12}], & A\,(\alpha_1-\alpha_2) \\ (\gamma_1\alpha_2-\gamma_2\alpha_1)(\beta_1-\beta_2), & [(\gamma_1\alpha_2)^{\frac12}-(\gamma_2\alpha_1)^{\frac12}]\,[\beta_1^{\frac12}-\beta_2^{\frac12}], & B\,(\beta_1-\beta_2) \\ (\alpha_1\beta_2-\alpha_2\beta_1)(\gamma_1-\gamma_2), & [(\alpha_1\beta_2)^{\frac12}-(\alpha_2\beta_1)^{\frac12}]\,[\gamma_1^{\frac12}-\gamma_2^{\frac12}], & C\,(\gamma_1-\gamma_2) \end{vmatrix} = 0,$$

and this is fulfilled since the sums of each of the columns vanish for

$$(\alpha_1-\alpha_2)\,(\beta_1\gamma_2-\beta_2\gamma_1)+\&c. = \begin{vmatrix} \alpha_1-\alpha_2, & \beta_1-\beta_2, & \gamma_1-\gamma_2 \\ \alpha_1, & \beta_1, & \gamma_1 \\ \alpha_2, & \beta_2, & \gamma_2 \end{vmatrix} = 0,$$

$$(\alpha_1^{\frac12}-\alpha_2^{\frac12})\,[(\beta_1\gamma_2)^{\frac12}-(\beta_2\gamma_1)^{\frac12}]+\&c. = \begin{vmatrix} \alpha_1^{\frac12}-\alpha_2^{\frac12}, & \beta_1^{\frac12}-\beta_2^{\frac12}, & \gamma_1^{\frac12}-\gamma_2^{\frac12} \\ \alpha_1^{\frac12}, & \beta_1^{\frac12}, & \gamma_1^{\frac12} \\ \alpha_2^{\frac12}, & \beta_2^{\frac12}, & \gamma_2^{\frac12} \end{vmatrix} = 0,$$

and $A\alpha_1+B\beta_1+C\gamma_1 = \text{contact} = A\alpha_2+B\beta_2+C\gamma_2$.

1640. (By S. A. RENSHAW, M.A.)—Find the locus of a point P, such that PA±PB = PC±PD, when A, B, C, D, P are points (1) in a plane, (2) in space.

Solution by ASÛTOSH MUKHOPÂDHYÂY.

1. To investigate the loci in question, we remark that, if four quantities X, Y, Z, U be connected by the equation $X^{\frac12}\pm Y^{\frac12}=Z^{\frac12}\pm U^{\frac12}$, they are in-

volved in the rational equation, easily obtained therefrom by transposition and squaring, $(X^2+Y^2+Z^2+U^2-2\Sigma XY)^2 = 64XYZU$,
which is, *in general*, of the fourth degree in X, Y, Z, U.

In the plane problem, take the line joining AB as the axis of x, and a line at right angles to it through A as the axis of y; then the points A, B, C, D are $(0, 0)$, $(\alpha_1, 0)$, (α_2, β_2), (α_3, β_3); and, if P be (x, y), the locus of P is

$$(x^2+y^2)^{\frac{1}{2}} \pm [(x-\alpha_1)^2+y^2]^{\frac{1}{2}} = [(x-\alpha_2)^2+(y-\beta_2)^2]^{\frac{1}{2}} \pm [(x-\alpha_3)^2+(y-\beta_3)]^{\frac{1}{2}} \quad \text{................(1)}.$$

Now, observing that the expressions under the radicals are of the second degree, and attending to the lamina, it is easy to see that this reduces to an equation of the eighth degree, which represents the locus required.

2. The extension to three dimensions is easy enough. Take the plane through A, B, C as the plane of xy, and θ as origin; let D be $(\alpha_3, \beta_3, \gamma_3)$, and P, (x, y, z); then it is obvious that the locus will be obtained by adding to the expressions under radicals in (1) some *quadratic* function of z. That equation, being simplified, will be found to be, as before, of the eighth degree in x, y, z; therefore the locus is *in general* an octic.

No simplification is introduced by taking oblique axes, the axes being the lines joining A and B, C and D.

[For the question itself, see Vol. IV., p. xvi., and for solutions of analogous questions (2718, 2737), see Vol. X., pp. 106, 107.]

3926. (By the Editor.)—A circle, whose radius is one foot, rolls from one end to the other on the *outside* of a quadrant of a circle whose radius is four feet, and then back again on the *inside* to its former position; show the form, and find the length and area of the closed curve described by that point in the rolling circle which was in contact with the quadrant at the commencement of the motion.

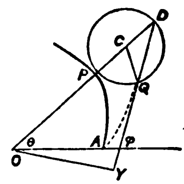
Solution by Prof. Evans, M.A.; Prof. Matz, M.A.; *and others.*

According to ordinary notation, if a and b be radii, equation to epicycloid is given by $x = (a+b)\cos\theta - b\cos\dfrac{a+b}{b}\theta$, $y = (a+b)\sin\theta - b\sin\dfrac{a+b}{b}\theta$,

where $$x = 5\cos\theta - \cos 5\theta, \quad y = 5\sin\theta - \sin 5\theta.$$

$$\text{Area} = \frac{1}{2}\int_0^{\frac{1}{2}\pi}(x\,dy - y\,dx) = \frac{5}{2}\int_0^{\frac{1}{2}\pi}[5(\cos\theta+\sin\theta) - b(\cos 5\theta + \sin 5\theta) + 5]\,d\theta$$
$$= (\tfrac{5}{2})^2\pi.$$

Area of hypocycloid is got by writing $-b$ for b, or in this case 3 for 5; hence area of hypocycloid $= (\frac{3}{2})^2\pi$, and whole area $= \frac{34}{4}\pi = \frac{17}{2}\pi$.



As, with the usual notation, ϕ = the angle of deflection, and AQ = s, we have

$$\phi-\theta = \frac{a\theta}{2b},\quad p = (a+2b)\sin\frac{a\theta}{2b} = (a+2b)\sin\frac{a\phi}{a+2b};$$

therefore $$\frac{ds}{d\phi} = p + \frac{d^2p}{d\phi^2} = \frac{4b\,(a+b)}{a+2b}\sin\frac{a\phi}{a+2b};$$

therefore $$s = \frac{4b}{a}(a+b)\left(1-\cos\frac{a\phi}{a+2b}\right).$$

The limits of ϕ are π to 0, therefore

$$s = \frac{8b\,(a+b)}{a}\left\{1-\cos\frac{a\pi}{a+2b}\right\} = \frac{4}{4}(5)\left\{1-\cos\frac{2\pi}{3}\right\} = \frac{5}{2} \text{ for epicycloid.}$$

Hypocycloid $= 3\,[1-\cos 2\pi] = 3$; therefore whole $= \frac{11}{2}$ ft.

7898. (By R. A. Roberts, M.A.)—A variable circular cylinder circumscribes a fixed tetrahedron. Show that the locus of a line drawn through a fixed point parallel to its edges is a cubic cone containing the six parallels to the edges of the tetrahedron.

Solution by J. P. Johnstone; Professor Chakravarti, M.A.; *and others.*

Taking the fixed point as origin, let the coordinates of the vertices of the tetrahedron be $(x_1y_1z_1)$, $(x_2y_2z_2)$, $(x_3y_3z_3)$, $(x_4y_4z_4)$ respectively, and let l, m, n be the direction cosines of the axis of a circumscribing circular cylinder. The lines passing through the vertices, whose direction cosines are l, m, n, will lie on a circular cylinder, if they pass through the intersection of the plane whose equation is (1) with the sphere whose equation is (2), $\quad lx+my+nz = 0,\quad x^2+y^2+z^2+2ax+2by+2cz+d = 0 \ldots\ldots(1, 2)$, where a, b, c are connected by the relation $la+mb+nc=0$, *i.e.*, the centre of the sphere lies in the plane.

Solving between the equations $\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$, of one of these lines and equation (1), we get

$x = x_1 - L_1 l,\ y = y_1 - L_1 m,\ z = z_1 - L_1 n,$ where $L_1 = l_1x + m_1y + n_1z$. Substituting these values for xyz in (2) and remembering $la+mb+nc=0$, we get $r_1^2 - L_1^2 + 2ax_1 + 2by_1 + 2cz_1 + d = 0$, where $r_1^2 = x_1^2+y_1^2+z_1^2$. Eliminating a, b, c, d between this, the three similar equations, and $la+mb+nc = 0$, we get

$$\begin{vmatrix} r_1^2-L_1^2, & x_1, & y_1, & z_1, & 1 \\ r_2^2-L_2^2, & x_2, & y_2, & z_2, & 1 \\ r_3^2-L_3^2, & x_3, & y_3, & z_3, & 1 \\ r_4^2-L_4^2, & x_4, & y_4, & z_4, & 1 \\ 0 & l & m & n & 0 \end{vmatrix} = 0.$$

Multiplying r_1^2, &c., by $l^2+m^2+n^2=1$, we have a homogeneous relation of the third degree in l, m, n, which represents a cubic cone. If we subtract the second row from the first and substitute for x_1-x_2, y_1-y_2, z_1-z_2, proportionals to l, m, n, we put the first row the same as the last, therefore the edges of the tetrahedron are parallel to generators of the cone.

7789. (By R. Tucker, M.A.)—AD is the bisector of the angle A of the triangle ABC; ω_1, ω are the Brocard angles of the triangles ABD, ABC: prove that $\Sigma_1^r \cot\omega_r - 4\cot\omega = (ab+bc+ca)/\Delta$, the summation being taken over the six triangles ABD, ACD, &c.

Solution by B. Hanumanta Rau, M.A.; J. O'Regan; *and others.*

Let $\angle ADB = \theta$, and let ω_1, ω_2 be the Brocard angles of the triangles ABD, ACD; then we have $\cot\omega = \cot A + \cot B + \cot C$,

$\cot\omega_1 = \cot\tfrac{1}{2}A + \cot B + \cot\theta, \quad \cot\omega_2 = \cot\tfrac{1}{2}A + \cot C + \cot(\pi-\theta),$

$\therefore\ \cot\omega_1 + \cot\omega_2 = 2\cot\tfrac{1}{2}A + \cot B + \cot C = 2\,\mathrm{cosec}\,A + \cot A + \cot\omega;$

$\therefore\ \Sigma_1^r \cot\omega_r = 2(\mathrm{cosec}\,A + \mathrm{cosec}\,B + \mathrm{cosec}\,C) + \cot A + \cot B + \cot C + 3\cot\omega,$

therefore $\quad \Sigma_1^r \cot\omega_r - 4\cot\omega = 2(\mathrm{cosec}\,A + \mathrm{cosec}\,B + \mathrm{cosec}\,C)$

$$= \frac{bc}{\Delta} + \frac{ca}{\Delta} + \frac{ab}{\Delta} = \frac{bc+ca+ab}{\Delta}.$$

7805. (By Professor Sylvester, F.R.S.)—If I represents the determinant $\begin{matrix} a & b & c \\ b & d & e \\ c & e & f \end{matrix}$ and $F\lambda$ (a cubic function of λ) is $e^{\lambda(\delta_d+2\delta_c)}\,I$, show that there are two values of λ, say λ_1, λ_2, of the form $\frac{M}{a}$, $\frac{N}{f}$ such that $a^3F\lambda_1 = Q_1^2$, $f^3F\lambda_2 = Q_2^2$, M, N, Q_1, Q_2 being rational integer functions of a, b, c, d, e, f.

Solution by W. J. C. Sharp, M.A.

Since I is the discriminant of the expression

$$ax^2 + 2bxy + 2cxz + dy^2 + 2eyz + fz^2 \equiv U \text{ say},$$

therefore (by Taylor's theorem) $F(\lambda)$ is the discriminant of

$$ax^2 + 2bxy + 2(c+2\lambda) \quad + (d+\lambda)\,y^2 + 2eyz + fz^2.$$

Now $\quad (ax+by+cz)^2 - aU \equiv (b^2-ad)\,y^2 + 2(bc-ae)\,yz + (c^2-af)\,z^2,$

and this is the product of two rational factors, if, and only if, $-aI$ be a perfect square (and of two real factors, if, and only if, $-aI$ be positive, so

that this is the condition that the tangents from $y = 0$, $z = 0$ to $U = 0$ should be real). Hence $-aF(\lambda)$ is a perfect square when, and only when,

$$[b^2 - a(d+\lambda)]\,y^2 + 2\,[b(c+2\lambda) - ae]\,yz + [(c+2\lambda)^2 - af]\,z^2,$$

or
$$[by + (c+2\lambda)\,z]^2 - a\,[(d+\lambda)\,y^2 + 2eyz + fz^2]$$

is a product of rational factors, *i.e.*, if it can be expressed as a difference of squares; therefore $f(d+\lambda) = e^2$ and $\lambda = \dfrac{e^2 - fd}{f} = \lambda_2$.

Then $-aF(\lambda_2)$ is the discriminant of

$$\left(b^2 - a\frac{e^2}{f}\right) y^2 + 2\left(\frac{b}{f}(cf + 2e^2 - 2fd) - ae\right) yz + \left(\frac{1}{f^2}(cf + 2e^2 - 2fd)^2 - af\right) z^2,$$

or
$$\frac{a}{f^3}\,[e^2(cf + 2e^2 - 2fd)^2 - 2bef^2(cf + 2e^2 - 2fd) + b^2f^4]$$

$$= \frac{a}{f^3}\,[cef + 2e^3 - 2efd - bf^2]^2,$$

and therefore $\qquad f^3F(\lambda_2) = -\,[cef + 2e^3 - 2efd - bf^2]^2,$

by symmetry, $\qquad a^3F(\lambda_1) = -\,[abc + 2b^3 - 2abd - a^2e]^2,$

where $\qquad \lambda_1 = \dfrac{b^2 - ad}{a}.$

7509. (By Professor Wolstenholme, M.A., Sc.D.)—In any tetrahedron ABCD, if s_1, s_2, s_3, s_4 be the sums of the lengths of the edges respectively meeting in A, B, C, D, and S_1, S_2, S_3, S_4 the sums of the dihedral angles at the same points; prove that, if $s_1 > s_2 > s_3 > s_4$, then $S_4 > S_3 > S_2 > S_1$.

Solution by W. J. C. Sharp, M.A.

Let ABCD be the tetrahedron, and let

DBC $= a_2$, BCD $= a_3$, CDB $= a_4$; DAC $= b_1$, ACD $= b_3$, CDA $= b_4$;
BAD $= c_1$, ABD $= c_2$, ADB $= c_4$; BAC $= d_1$, ABC $= d_2$, ACB $= d_3$;
and B_1, C_1, D_1; C_2, D_2, A_2; D_3, A_3, B_3; A_4, B_4, C_4 the angles of the spherical triangles determined by the solid angles, so that

$$B_1 = A_2,\ C_1 = A_3,\ D_1 = A_4,\ C_2 = B_3,\ D_2 = B_4,\ D_3 = C_4,$$

and $\qquad a_2 + a_3 + a_4 = b_3 + b_4 + b_1 = c_4 + c_1 + c_2 = d_1 + d_2 + d_3 = \pi.$

Now, if E_1 be the spherical excess of the spherical triangle whose sides are b_1, c_1, d_1, and E_2 that of the one whose sides are c_2, d_2, a_2,

$$\cos\tfrac{1}{2}S_1 = \sin\tfrac{1}{2}E_1 = \frac{\sin\frac{1}{2}c_1 \sin\frac{1}{2}d_1}{\cos\frac{1}{2}b_1}\sin B_1,$$

$$\cos\tfrac{1}{2}S_2 = \sin\tfrac{1}{2}E_2 = \frac{\sin\frac{1}{2}c_2 \sin\frac{1}{2}d_2}{\cos\frac{1}{2}a_2}\sin A_2.$$

Hence $S_1 < = > S_2$ according as

$$\sin^2\tfrac{1}{2}c_1 \sin^2\tfrac{1}{2}d_1 \cos^2\tfrac{1}{2}a_2 > = < \sin^2\tfrac{1}{2}c_2 \sin^2\tfrac{1}{2}d_2 \cos^2\tfrac{1}{2}b_1.$$

Now, by Plane Trigonometry,

$$\sin^2 \tfrac{1}{2}c_1 : \sin^2 \tfrac{1}{2}c_2 = \frac{AB+BD-AD}{AD} : \frac{AD+AB-BD}{BD},$$

and
$$\sin^2 \tfrac{1}{2}d_1 : \sin^2 \tfrac{1}{2}d_2 = \frac{AB+BC-CA}{AC} : \frac{AB+AC-BC}{BC},$$

$$\cos^2 \tfrac{1}{2}a_2 = \frac{(BD+DC+BC)(BD+BC-DC)}{BD \, . \, BC},$$

$$\cos^2 \tfrac{1}{2}b_1 = \frac{(AC+CD+DA)(AC+DA-DC)}{AD \, . \, AC}.$$

Therefore $S_1 < = > S_2$, according as

$(AB+BD-AD)(AB+BC-AC)(BD+DC+BC)(BD+BC-DC)$ is $> = <$

$(AD+AB-BD)(AB+AC-BC)(AC+CD+DA)(AC+DA-DC)$,

according as $(s_2-AD-BC)(s_2-AC-BD)(s_2+DC-AB)(s_2-DC-AB)$

is $> = < (s_1-BD-AC)(s_1-AD-BC)(s_1+DC-AB)(s_1-AB-DC)$,

that is, according as $s_2 > = < s_1$, and hence by symmetry the proposition follows (since each of the factors on each side of the inequality are positive).

[The PROPOSER remarks that he has not yet been able to discover the law for σ_1, σ_2, σ_3, σ_4, the sums of the plane angles at A, B, C, D, but has found that they are *not* always in the same order of magnitude as S_1, S_2, S_3, S_4, which indeed follows from the theorem that when $s_1 = s_2$, then $S_1 = S_2$, since σ_1 is not then $= \sigma_2$. At the same time (excluding cases where $s_1 = s_2$), he has only found six out of about 150 in which the order of magnitude of the σ's is different from that of the S's. In each of these cases, two of the σ's are nearly equal and the two corresponding S's also nearly equal, and the orders of magnitude differ by one displacement: such as $S_1 > S_2 > S_3 > S_4$, $\sigma_1 > \sigma_3 > \sigma_2 > \sigma_4$, where S_2, S_3 are nearly equal, and also σ_2, σ_3.]

7778. (By Professor HUDSON, M.A.) — A Galileo's and a common telescope have the same object-glass, and their eye-glasses have equal focal lengths; also the uniformly bright field is of the same extent in both: prove that the diameter of the stop in the common telescope should be half the difference of the breadths of the eye-glasses.

Solution by B. HANUMANTA RAU, M.A.; SARAH MARKS; *and others.*

Let F, f be the focal lengths of the object and eye-glasses, and x, b, a, a' the half-breadths of the stop, object-glass, and eye-glasses; then in the astronomical telescope

$$a-x : f \;\; b+x : F, \quad \text{therefore} \quad x = \frac{Fa-fb}{F+f},$$

therefore angular radius of field of view $= \frac{x}{f} = \frac{Fa - fb}{f(F+f)}$ ....(1),

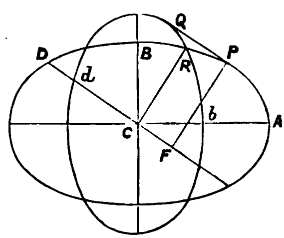
$= $ field of view in Galileo's $= \frac{fb - Fa'}{f(F-f)}$ ....(2).

From (1) and (2), $\frac{x}{f} = \frac{Fa - Fa'}{2fF} = \frac{a-a'}{2f}$,

therefore $x = a - a'$, or $2x = 2a - 2a'$.

3372. (By Professor GENESE, M.A.)—Two similar ellipses are placed so that the major axis of either coincides with the minor of the other; prove that the lines joining the common centre to the common points are perpendicular to the common tangents.

Solution by the Rev. T. C. SIMMONS, M.A.



Let the semi-axes be (CA, CB), (Ca, Cb), where CA : Ca = CB : Cb = m, and let the diameter CdD be perpendicular to the common chord CR; then, since CR makes with CA the same angle as Cd makes with Ca, and the ellipses are similar,

CR : CA = Cd : Ca or CR = m . Cd;

and in a similar manner we obtain CD = m . CR; hence CR² = CD . Cd; also CD = m^2 . Cd. Draw now the common tangent PQ and the parallel diameter $Cd'D'$ to which PF is perpendicular; then PF . CD′ = AC . BC and PF . Cd' = aC . bC, whence by division CD′ = m^2 . Cd', therefore CD : Cd = CD′ : Cd', *i.e.*, $CD'd'$ coincides with CDd, whence it follows that CR is at right angles to PQ.

7885. (By J. BRILL, B.A.)—If ABCDE be any pentagon inscribed in a circle, prove that

$$EA^2 . BC . CD . BD + EC^2 . AB . BD . AD = EB^2 . AC . CD . AD + ED^2 . AB . BC . AC.$$

Solution by ÂSÛTOSH MUKHOPÂDHYÂY.

The theorem holds for any pentagon *four* of whose vertices lie on a circle, as is evident at once from the following theorem, due to Dr.

SALMON (*Conics*, § 94), which has been extended in the *Messenger of Mathematics*, Vol. XIII., pp. 157 160:—

"If A, B, C, D be any four points on a circle, and E any fifth point taken arbitrarily, then $EA^2 . BCD + EC^2 . ABD = EB^2 . ACD + ED^2 . ABC$, when BCD denotes the area of the triangle BCD."

Now, since $BCD = \frac{1}{2} BC . CD . \sin C$, $ABD = \frac{1}{2} AB . AD . \sin A$, &c., this may be written,

$$EA^2 . BC . CD . \sin C + EC^2 . AB . AD . \sin A$$
$$= EB^2 . AD . DC . \sin D + ED^2 . AB . BC . \sin B \text{.........(1)}.$$

Again, we have, in any circle, chord = diameter × sin (angle subtended at circumference). Therefore, if d be the diameter of the circle, we get

$$BD = d . \sin A, \quad AC = d . \sin D;$$

hence, remembering that $\sin C = \sin A$, and $\sin B = \sin D$ (since the opposite angles are supplementary), we have the relation,

$$\frac{BD}{\sin C} = \frac{BD}{\sin A} = \frac{AC}{\sin D} = \frac{AC}{\sin B},$$

and, substituting for $\sin A$, $\sin B$, &c., in (1), we have the identity in question.

7958. (By Rev. T. R. TERRY, M.A.)—Solve (1) the equation

$$w_{x+2} = \left[\tfrac{5}{2} + (-1)^x \tfrac{3}{2}\right] w_{x+1} - w_x;$$

and hence (2) show that, if u_x and v_x both satisfy this equation, and if $u_1 = 1$, $v_1 = 1$, $u_2 = 4$, $v_2 = 3$, then $(x+1)\, u_x = 2xv_x$.

Solution by R. KNOWLES, B.A.; NILKANTA SARKAR, B.A.; *and others.*

If $w_x = w'_x \left[\tfrac{5}{2} + (-1)^x \tfrac{3}{2}\right]^{\frac{1}{2}}$, then (HYMER's *Calc.*, p. 66),

$w'_{x+2} - 2w'_{x+1} + w'_x = 0$, and the roots of $m^2 - 2m + 1 = 0$ are each $= 1$.

$\therefore\ w_x = (c_1 + c_2 x)\left[\tfrac{5}{2} + (-1)^x \tfrac{3}{2}\right]^{\frac{1}{2}}$, but $u_1 = 1 = c_1 + c_2$, $u_2 = 4 = 2(c_1 + 2c_2)$, therefore $\quad u_x = x\left[\tfrac{5}{2} + (-1)^x \tfrac{3}{2}\right]^{\frac{1}{2}}$.

Similarly, from $v_1 = 1$, $v_2 = 3$, we have

$$v_x = \tfrac{1}{2}(1+x)\left[\tfrac{5}{2} + (-1)^x \tfrac{3}{2}\right]^{\frac{1}{2}}, \text{ therefore } (x+1)\, u_x = 2x\, v_x.$$

7785. (By Dr. CURTIS.)—If a triangular area be so sunk in a homogeneous liquid, that its Centre of Pressure coincide with the intersection of the three lines got by joining the mid-point of each side with the mid-point of the perpendicular let fall on it from the opposite angle; prove that, H_1, H_2, H_3 being the depths to which the mid-points of the sides a, b, c are immersed, $H_1 : H_2 : H_3 = \cot A : \cot B : \cot C$.

Solution by Rev. T. C. SIMMONS, M.A.; *and* B. HANUMANTA RAU, M.A.

The lines joining the mid-point of each side with the mid-point of the

perpendicular on it from the opposite angle, meet at a point whose distances x, y, z from the sides are such that $\frac{x}{a} = \frac{y}{b} = \frac{z}{c} \equiv k$; and by hypothesis $2H_1 = h_2 + h_3$, $2H_2 = h_3 + h_1$, $2H_3 = h_1 + h_2$, therefore

$$H_1 + H_2 + H_3 = h_1 + h_2 + h_3;$$

hence, substituting these values in the expressions for x, y, z, given in the solution to Quest. 7706 (Vol. 42, p. 21), we have

$$\frac{2(H_2+H_3)}{H_1+H_2+H_3} = 4\frac{x}{p_1} = 4k\frac{a}{p_1} = 4k(\cot B + \cot C),$$

$$\therefore \quad \frac{H_2+H_3}{\cot B + \cot C} = 2k(H_1+H_2+H_3) = \frac{H_1+H_2}{\cot A + \cot B} = \frac{H_1+H_3}{\cot A + \cot C}.$$

Adding two numerators and subtracting the third gives the result.

7877. (By H. L. Orchard, B.Sc., M.A.)—A heavy particle is projected with unit-velocity, in a direction of 45° with the horizon. Find when the radius of curvature of the path will be unity.

Solution by Â. Mukhopâdhyây; Rev. T. C. Simmons, M.A.; *and others.*

Taking the point of projection as origin, let x, y be the horizontal and vertical coordinates of the particle at any time; then $y = x - gx^2$,

therefore $$\frac{dy}{dx} = 1 - 2gx, \quad \frac{d^2y}{dx^2} = -2g;$$

therefore $$\rho = \pm\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}} \Big/ \frac{d^2y}{dx^2} = \pm\sec^3\theta \Big/ \frac{d^2y}{dx^2} = \pm\frac{\sec^3\theta}{2g},$$

where θ is the angle which the direction of the particle's motion makes with the horizon at the time; hence ρ numerically $= 1$ when $\sec^3\theta = 2g$. If a foot and a second are taken as units, this gives $\cos\theta = \frac{1}{4}$ nearly, or $\theta = -75° 25'$ nearly, and the required point lies *below* the horizontal plane through the point of projection, and on the other side of the parabola.

1194 & 4009. (By the Editor.)—(1194.)—If P be a point in the plane of a triangle ABC; α, β, γ the angles BPC, CPA, APB; a, b, c the sides of the triangle; and x, y, z the lines PA, PB, PC: show that

$$\frac{\sin^2(\alpha - A)}{x^2} = \frac{\sin^2\beta}{b^2} + \frac{\sin^2\gamma}{c^2} \pm \frac{2\sin\beta\sin\gamma\cos(\alpha - A)}{bc},$$

$$\frac{\sin^2(\beta - B)}{y^2} = \frac{\sin^2\alpha}{a^2} + \frac{\sin^2\gamma}{c^2} \pm \frac{2\sin\alpha\sin\gamma\cos(\beta - B)}{ac},$$

$$\frac{\sin^2(\gamma - C)}{y^2} = \frac{\sin^2\alpha}{a^2} + \frac{\sin^2\beta}{b^2} \pm \frac{2\sin\alpha\sin\beta\cos(\gamma - C)}{ab},$$

+ or − according as P is inside or outside the triangle ABC.

(4009.) **Show that the values of x, y, z, from the equations**
$4x^2-2xy+4y^2=81$, $8x^2+11xz+8z^2=242$, $4y^2+7yz+4z^2=100$, are

$$\frac{51\sqrt{15}+420\sqrt{2}}{23910+2700\sqrt{30}},\quad \frac{45\sqrt{15}+330\sqrt{2}}{23910+2700\sqrt{}},\quad \frac{140\sqrt{15}-120\sqrt{2}}{\sqrt{23910+2700\sqrt{30}}},$$

or, in decimals, 4·02345674, 3·25832046, 1·89362154.

Solution by D. Biddle; Belle Easton; *and others.*

(1194.) We have

$$\sin^2(\alpha-A)=\sin^2(ABP+ACP)=(\sin ABP\,.\,\cos ACP+\sin ACP\,.\,\cos ABP)^2$$
$$=\sin^2 ABP+\sin^2 ACP-2\sin^2 ABP\,.\,\sin^2 ACP$$
$$+2\sin ABP\,.\,\sin ACP\,.\,\cos ABP\,.\,\cos ACP;$$

but $\cos(\alpha-A)=\cos(ABP+ACP)=\cos ABP\,.\,\cos ACP-\sin ABP\,.\,\sin ACP$,

$\therefore\ \sin^2(\alpha-A)=\sin^2 ABP+\sin^2 ACP+2\sin ABP\sin ACP\cos(\alpha-A)$.

Now $\sin ABP : x=\sin\gamma : c$, and $\sin ACP : x=\sin\beta : b$,

therefore $\sin^2(\alpha-A)=\dfrac{x^2\sin^2\gamma}{c^2}+\dfrac{x^2\sin^2\beta}{b^2}+\dfrac{2x^2\sin\gamma\sin\beta\cos(\alpha-A)}{}$;

whence we obtain the first equation in the question; and the other two equations follow in the same way.

[If P be an *internal* point, and $ABP=\theta$, $ACP=\phi$, we have $\theta+\phi=\alpha-A$; also $\sin^2(\theta+\phi)=\sin^2\theta+\sin^2\phi+2\sin\theta\sin\phi\cos(\theta+\phi)$, and from the triangles APC, APB, $\dfrac{\sin\phi}{x}=\dfrac{\sin\beta}{b}$, $\dfrac{\sin\theta}{x}=\dfrac{\sin\gamma}{c}$,

therefore $\dfrac{\sin^2(\alpha-A)}{x^2}=\dfrac{\sin^2\beta}{b^2}+\dfrac{\sin^2\gamma}{c^2}+2\dfrac{\sin\beta}{b}\dfrac{\sin\gamma}{c}\cos(\alpha-A)$.

The other two formulæ follow by symmetry. If P be an *external* point, it may be readily shown that the formulæ have, as stated in the Question, the last term of each negative.]

(4009.) Writing the equations in the form

$x^2-2\,.\,\frac{1}{4}xy+y^2=\frac{81}{4}$, $x^2+2\,.\,\frac{11}{16}xz+z^2=\frac{121}{4}$, $y^2+2\,.\,\frac{7}{8}yz+z^2=25\ldots(1,2,3)$,

it will be found that their solution may be at once deduced from the formulæ in (1194); for, from the relations of the figure, we have

$$x^2+y^2-2\cos\gamma\,.\,xy=c^2,\quad x^2+z^2-2\cos\beta\,.\,xz=b^2\ldots\ldots\ldots(4,5),$$
$$y^2+z^2-2\cos\alpha\,.\,yz=a^2\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6).$$

We have also $\alpha+\beta+\gamma=360°$; thus, comparing (1), (2), (3) with (4), (5), (6), it will be found that, if $-\frac{11}{16}$ and $-\frac{7}{8}$ be the cosines of two angles, then $\frac{1}{4}$ will be the cosine of 360° minus their sum.

Hence, to apply the formula in (4009) to the equations (1), (2), (3), we shall have the following relations :—

$$\cos\alpha=-\tfrac{7}{8},\quad \cos\beta=-\tfrac{11}{16},\quad \cos\gamma=\tfrac{1}{4},\quad \sin\alpha=\tfrac{1}{8}\sqrt{15},$$
$$\sin\beta=\tfrac{3}{16}\sqrt{15},\quad \sin\gamma=\tfrac{1}{4}\sqrt{15};\quad \text{also,}\quad a=5,\quad b=\tfrac{11}{2},\quad c=\tfrac{9}{2};$$

whence $\cos A=\frac{17}{33}$, $\cos B=\frac{1}{3}$, $\cos C=\frac{7}{11}$,

$$\sin A=\tfrac{20}{33}\sqrt{2},\quad \sin B=\tfrac{2}{3}\sqrt{2},\quad \sin C=\tfrac{6}{11}\sqrt{2};$$

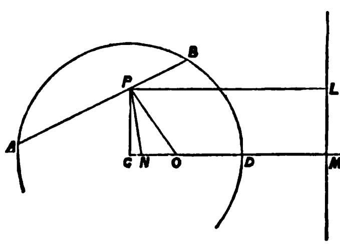
and therefrom we readily find

$$\cos(\alpha-A) = \tfrac{1}{384}(20\sqrt{30}-119),\quad \sin(\alpha-A) = \tfrac{1}{384}(17\sqrt{15}-140\sqrt{2}),$$
$$\cos(\beta-B) = \tfrac{1}{48}(6\sqrt{30}-11),\quad \sin(\beta-B) = \tfrac{1}{48}(3\sqrt{15}+22\sqrt{2}),$$
$$\cos(\gamma-C) = \tfrac{1}{24}(6\sqrt{30}+7),\quad \sin(\gamma-C) = \tfrac{1}{24}(7\sqrt{15}-6\sqrt{2}).$$

By substituting these results in (1194), we obtain the result stated.

7765. (By W. J. McClelland, B.A.)—Prove that, for any point P on a chord AB of a circle, AP . BP + OP² = 2CO . PL, where C is the centre of the circle, O the limiting point, and L the radical axis.

Solution by J. Brill, B.A.; Â. Mukhopâdhyây; *and others.*



Draw PN perpendicular to CO; then we have

$$AP.PB + OP^2 = CD^2 - CP^2 + OP^2$$
$$= (CM^2 - MO^2) + CO^2 - 2CN.CO$$
$$= CO(CM + MO + CO - 2CN)$$
$$= 2CO.NM = 2CO.PL.$$

7620. (By Rev. T. C. Simmons, M.A.)—If A, B, C, D, E, F are six collinear points such that the three ranges ACDE, ABCE, ACEF are all harmonic, show that the ranges ABDF, BCDF, BDEF are also harmonic.

Solution by B. Hanumanta Rau, M.A.; N. Sarkar, M.A.; *and others.*

Let AB, AC, AD, AE, and AF $= b, c, d, e, f$; then

$$\frac{1}{c}+\frac{1}{e}=\frac{2}{d},\quad \frac{1}{b}+\frac{1}{e}=\frac{2}{c},\quad\text{and}\quad \frac{1}{c}+\frac{1}{f}=\frac{2}{e}\ \ \text{......(1, 2, 3).}$$

Adding, $\frac{1}{b}+\frac{1}{f}=\frac{2}{d}$, *i.e.*, ABDF is an harmonic range ....(4).

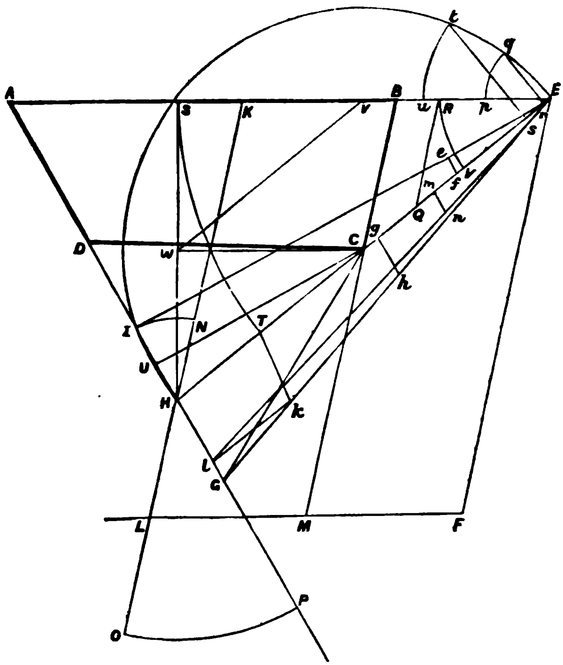
Again, from (1) and (2) $\frac{c-b}{bc} = 2\frac{d-c}{cd}$,

and from (4) $\frac{f-b}{bf} = 2\frac{f-d}{fd}$, whence $\frac{c-b}{f-b} = \frac{d-c}{f-d}$,

that is, BCDF is harmonic. Similarly for the third range.

2931. (By the EDITOR.)—**Construct a quadrilateral geometrically, having given the angles A, B, and the sums of the sides $a+b$, $b+c$, $c+d$.**

Solution by D. BIDDLE.



Let $AE = a+b$, $EF = b+c$, $AG = c+d$, $\angle EAG = A$, $AEF = B$. Then it is evident that C must be on EH, which bisects AEF.

Let EH be unity, and on it describe the semi-circle HItqE, join EI or draw it perpendicular to AG, and through H draw KO parallel to EF; also draw FL parallel to AE, and make LO = HL, and, producing AG, make HP = HO; also make HN = HI, and EQ = 2HI. Then draw QR parallel to HK, and make EV = ER (= QR); also ES = HV, and ET = ES. Join EG, and draw Tk parallel to AP, and kl parallel to EH; also join El, and making Em = GP (= 2HL − HG), draw mn parallel to AP. Also make Eg = NK and draw gh parallel to AP, and make Ee = HL, and draw ef parallel to AP. Next, make Ep = $ef + gh$, and with E as centre describe the arc pq, cutting the semi-circle in q; then draw qr perpendicular to EH, and make $rs = mn$. Draw st at right angles with EH, to meet the semi-circle in t, and again with E as centre describe the arc tu; also make uv = Ep (= $ef+gh$). Finally, join HS, and draw vw, wC parallel to EH, AE respectively. C will be the point required in EH,

and by drawing BM through it, parallel to EF, and making $CD = CM$ ($= EF - BC$), we make $AD + DC = AG$, and the quadrilateral is constructed upon the given conditions.

For, retracing the foregoing steps—we easily see that

$$EH\ (=1) : HC = ES : wC = ET : Ev = HV : Eu + Ep$$
$$= 1 - 2HI . HK : Es^{\frac{1}{2}} + ef + gh;$$

and, since $\quad Es = Er + mn = Ep^2 + mn = (ef + gh)^2 + mn,$

therefore $\quad 1 : HC = 1 - 2HI . HK : [(ef + gh)^2 + mn]^{\frac{1}{2}} + ef + gh$

$$= 1 - 2HI . HK : [\{(EF - HK) HI + (HK - HI) HG\}^2$$
$$+ HG \{2 (EF - HK) - HG\} (1 - 2HI . HK)]^{\frac{1}{2}}$$
$$+ (EF - HK) HI + (HK - HI) HG,$$

which is the exact ratio obtained by observing that

$$CD = DG = CM = EF - BC,$$

and that accordingly $GC^2 = 2GU . CM$, which resolves itself into

$$HC^2 + HG^2 + 2HG . HI . HC = 2 (HG + HI . HC)(EF - BC)$$
$$= 2 (HG + HI . HC) [EF - (1 - HC) HK],$$

whence $\quad 1 : HC = 1 - 2HI . HK : \&c.$

On the Application of Joachimstahl's Method of Studying Surfaces, and of an Extension of it to Surfaces Defined by Quaternion Equations, including the Solution of Question 7821.

By W. J. C. Sharp, M.A.

I. If q_1 and q_2 be the vectors of two points P and Q, and $q = \dfrac{\lambda q_1 + \mu q_2}{\lambda + \mu}$; q is the vector of the mean centre of λ at P and μ at Q, and therefore of the point in PQ where it is divided in the ratio of $\mu : \lambda$.

Similarly, if q_1, q_2, and q_3 be the vectors of three points P, Q, and R, and $q = \dfrac{\lambda q_1 + \mu q_2 + \nu q_3}{\lambda + \mu + \nu}$; q is the vector of the mean centre of λ at P, μ at Q, and ν at R, *i.e.*, of that point in the plane of PQR of which the areal coordinates, referred to this triangle, are λ, μ, ν.

And if q_1, q_2, q_3, and q_4 be the vectors of four non-coplanar points P, Q, R, and S, and $q = \dfrac{\lambda q_1 + \mu q_2 + \nu q_3 + \pi q_4}{\lambda + \mu + \nu + \pi}$; q is the vector of the mean centre of λ at P, μ at Q, ν at R, and π at S; *i.e.*, of the point whose tetrahedral coordinates, referred to P, Q, R, S as tetrahedron of reference, are λ, μ, ν, π.

II. Let $S . q\phi q = 1$ represent a central quadric, ϕq being a self-conjugate linear and vector function of q (Tait's *Quaternions*, p. 173); and for q substitute $\dfrac{\lambda q_1 + \mu q_2}{\lambda + \mu}$.

$$\therefore \quad \lambda^2 (S . q_1\phi q_1 - 1) + 2\lambda\mu (S . q_1\phi q_2 - 1) + \mu^2 (S . q_2\phi q_2 - 1) = 0$$

is the equation which determines the ratios in which the line joining the two points q_1 and q_2 is cut by the surface.

Now, (i.) if q_1 be on the surface, one value of $\frac{\mu}{\lambda}$ is zero, and the other is obtained from $2\lambda(S.q_1\phi q_2-1)+\mu(S.q_2\phi q_2-1)=0$, and, that the second may vanish, it is necessary and sufficient that $S.q_1\phi q_2-1=0$.

But, when this is the case, q_2 is any point on the tangent plane at q_1, the equation to which is therefore $S.q_1\phi q=1$ or the equivalent $Sq\phi q_1=1$.

(ii.) If q_1 be not on the surface, the condition for contact is

$$(S.q_2\phi q_1-1)^2=(S.q_1\phi q_1-1)(S.q_2\phi q_2-1).$$

Consequently $(S.q\phi q_1-1)^2=(S.q_1\phi q_1-1)(S.q\phi q-1)$ is the equation to the system of tangent lines from q_1 to the surfaces, *i.e.*, to the tangent cone whose vertex is at q_1. Also, the line joining q_1 and q_2 is cut harmonically by the surface if $S.q_1\phi q_2-1=0$; that is to say, the fourth harmonic of the points in which any line through q_1 is cut by the surface lies on the plane $S.q_1\phi q-1=0$, or, what is the same thing, $S.q\phi q_1-1=0$, which therefore is the polar plane of q_1, and this meets the surface along the curve of contact of the tangent cone. From the form of the equation, the relation between q and q_1 is reciprocal.

(iii.) If the line joining q_1 and q_2 lie entirely on the surface, *i.e.*, if it be a generator, the equation must be satisfied by all values of $\mu:\lambda$, therefore $S.q_1\phi q_1=S.q_2\phi q_1=S.q_1\phi q_2=S.q_2\phi q_2=1$. Now let $q_2-q_1=x\pi$, so that π is a vector parallel to the line, then

$$\begin{aligned}x^2S.\pi\phi\pi &= S.(q_2-q_1)\phi(q_2-q_1)\\ &= S.q_2\phi q_2-S.q_2\phi q_1-S.q_1\phi q_2+S.q_1\phi q_1=0,\end{aligned}$$

and $S.q\phi q=0$ is the equation to a cone the generators of which are parallel to those of the surface and having its vertex at the centre, *i.e.*, to the asymptotic cone; as also appears by putting $q_1=0$ in the equation to the tangent cone.

III. If $\frac{\lambda q_1+\mu q_2+\nu q_3}{\lambda+\mu+\nu}$ be substituted for q in the same equation $S.q\phi q=1$, the resulting equation in λ, μ, ν will be the equation in areal coordinates to the section of the surface by the plane through q_1, q_2, q_3, those points being the angular points of the triangle of reference. The result is

$$\begin{aligned}&\lambda^2(S.q_1\phi q_1-1)+\mu^2(S.q_2\phi q_2-1)+\nu^2(S.q_3\phi q_3-1)\\ &\quad+2\mu\nu(S.q_2\phi q_3-1)+2\nu\lambda(S.q_3\phi q_1-1)+2\lambda\mu(S.q_1\phi q_2-1)=0;\end{aligned}$$

and, if this be the tangent plane at q_1, it reduces by the last article to

$$\mu^2(S.q_2\phi q_2-1)+\nu^2(S.q_3\phi q_3-1)+2\mu\nu(S.q_2\phi q_3-1)=0,$$

which represents two straight lines, the generators through q_1; and these are real and different, coincident or imaginary, according as

$$(S.q_2\phi q_3-1)^2-(S.q_2\phi q_2-1)(S.q_3\phi q_3-1)$$

is positive, zero, or negative.

The corresponding condition in the case of ordinary coordinates is, that the generators are real and distinct, real and coincident, or imaginary, according as the discriminant is positive, zero, or negative; and hence, if

$$S.q_1\phi q_1=1,\quad S.q_2\phi q_1=1,\quad \text{and}\quad S.q_3\phi q_1=1,$$

$$(S.q_2\phi q_3-1)^2-(S.q_2\phi q_2-1)(S.q_3\phi q_3-1)$$

is a quaternion equivalent of the discriminant.

The condition that the plane through q_1, q_2, q_3 should touch the surface is

$$\begin{vmatrix} S.q_1\phi q_1-1, & S.q_1\phi q_2-1, & S.q_1\phi q_3-1 \\ S.q_2\phi q_1-1, & S.q_2\phi q_2-1, & S.q_2\phi q_3-1 \\ S.q_3\phi q_1-1, & S.q_3\phi q_2-1, & S.q_3\phi q_3-1 \end{vmatrix} = 0,$$

the discriminant of the equation in λ, μ, ν.

IV. The equation in tetrahedral coordinates, referred to a tetrahedron of which the vertices are q_1, q_2, q_3, and q_4, is obtained by substituting $\dfrac{\lambda q_1+\mu q_2+\nu q_3+\pi q_4}{\lambda+\mu+\nu+\pi}$ for q in the equation to the surface; and hence it appears that, if the resulting equation only contain the squares of the tetrahedral coordinates, the following equations must hold:

$$S.q_1\phi q_2 = S.q_1\phi q_2 = S.q_1\phi q_3 = S.q_2\phi q_3 = S.q_2\phi q_4 = S.q_3\phi q_4 = 1,$$

and, by II., the tetrahedron must be self-conjugate.

If q_1, q_2, q_3 be so chosen as to be rectangular vectors of lengths a, b, and c, and bcx, cay, abz, and $abc\left(1-\dfrac{x}{a}-\dfrac{y}{b}-\dfrac{z}{c}\right)$ be substituted for λ, μ, ν, π in the resulting equation, this will give the rectangular equation to the quadric.

V. Similarly in the case of any surface, when the equation is reduced to the form $S.\nu q = 1$ (HAMILTON's *Lectures*, Art. 575), or given in any scalar form, the substitution $q = \dfrac{\lambda q_1+\mu q_2}{\lambda+\mu}$ will give the ratios in which the line joining q_1 and q_2 is cut by the surface, and as, when the surface is of the n^{th} order the equation in $\lambda:\mu$ is of that order, ν will be of the $(n-1)^{\text{th}}$ order in q; also the coefficients of the various powers of μ will, when equated to zero, be the equations to the successive polars of q_1, just as in ordinary Geometry. (SALMON's *Geometry of Three Dimensions*, p. 209.) Also, the substitution $q = \dfrac{\lambda q_1+\mu q_2+\nu q_3}{\lambda+\mu+\nu}$ will give the equation, in areal coordinates, to the section made by the plane through q_1, q_2, q_3, and, as before, it will appear at once that the point of contact is a double point on the curve of section by the tangent plane, and that consequently the condition, that the plane through q_1, q_2, q_3 should touch the surface, is that the discriminant of the equation in $\lambda:\mu$ should vanish, while the nature of its contact is determined by the nature of the node, and the two inflexional tangents are those to the plane curve at the node.

The substitutions of IV. will give the equation to the surface in tetrahedral or rectangular coordinates.

7618. (By C. LEUDESDORF, M.A.)—The triangle of reference being equilateral, prove that the envelope of the director-circles of the conic whose trilinear equation is $kx^{-1} = y^{-1}+z^{-1}$, for different values of k, is the curve

$$4yz(x+y+z)^2 = [y^2+yz+z^2+5(yz+zx+xy)][3(y^2+yz+z^2)-(yz+zx+xy)].$$

Solution by B. HANUMANTA RAU, M.A.; G. G. STORR, B.A.; *and others.*

The equation of the pair of tangents drawn from the point (x', y', z') to the conic $-kyz+xy+xz=0$, is

$$[x(y'+z')+y(x'-kz')+z(x'-ky')]^2 = 4(xy+xz-kyz)(x'y'+x'z'-ky'z').$$

The two lines represented by this equation will be at right angles, provided $(y'+z')^2+(x'-kz')^2+(x'-ky')^2+(4-2k)(x'y'+x'z'-ky'z')$

$$-(y'+z')(x'-kz')-(y'+z')(x'-ky')-(x'-kz')(x'-ky')=0.$$

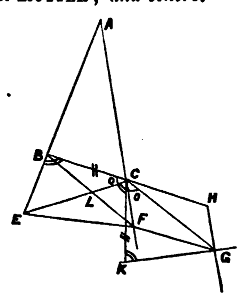
Suppressing the accents, the equation to the director-circle is

$$k^2(y^2+yz+z^2)-k(3xy+3xz+2yz-y^2-z^2)+(x+y+z)^2=0.$$

The envelope for different values of k is the same as the condition of equal roots of k; or $(3xy+3xz+2yz-y^2-z^2)^2 = 4(y^2+yz+z^2)(x+y+z)^2$, which is equivalent to Mr. LEUDESDORF's result.

7610. (By J. EDWARD, M.A., B.Sc.)—Draw a straight line EF terminated by the sides AB, AC of a triangle ABC, so as to make CE = EF = FB.

Solution by A. H. CURTIS, LL.D., D.Sc.; E. RUTTER; *and others.*



Suppose EF drawn as required; complete the parallelograms BFGC, CFGH; draw CG, make $\angle$KCB = GCE; take CK = CB, and draw KG. As EC = EF, $\angle$ECF = EFC, similarly, $\angle$EBF = BEF; hence

$$\angle\text{ECF}+\text{EBF} = \text{EFC}+\text{BEF},$$

and is therefore known; but $\angle$EBC and $\angle$FCB are known, therefore

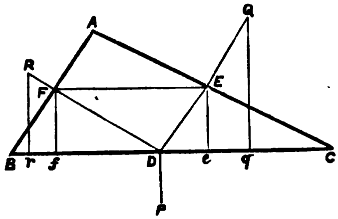
$$\angle\text{CLF} = \text{LBC}+\text{LCB}$$
$$= (\text{EBC}+\text{FCB})-(\text{EBF}+\text{ECF})$$

is known, or its supplement $\angle$ELF, or $\angle$GCE, or $\angle$KCB is known; the point K is therefore known, and comparing the triangles CBE, CKG, in which BC, CE, and contained angle = KC, CG and contained angle, $\angle$CKG = CBE, and therefore known, therefore KG, one locus of G, is known. Again CH = FG = BC; hence a second locus of G is HG parallel to AC through H, got by taking CH = BC: the intersection of these two loci gives G; inflect CE and then EF each = CG, and EF is found.

7865. (By Professor HUDSON, M.A.)—On the sides of any triangle similar regular polygons are described, and equal masses are placed at all the corners; prove that the centre of gravity of the masses coincides with that of the triangle.

Solution by Rev. T. C. SIMMONS, M.A.; N. SARKAR, M.A.; *and others.*

Let D, E, F be the mid-points of the sides, P, Q, R the centroids of the respective polygons: then we evidently require the C. G. of three equal masses at P, Q, R, where PD, QE, RF are perpendicular to the sides, and are equal respectively to μa, μb, μc. Take eq, fr the projections on BC of EQ, FR; then $eq = \text{EQ}\sin\text{C} = \mu b \sin\text{C} = \mu c \sin\text{B} = fr$, therefore



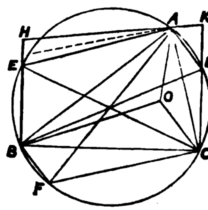
$$\text{B}q + \text{B}r = \text{B}e + \text{B}f \text{ or } \text{BD} + \text{B}q + \text{B}r = \text{BD} + \text{B}e + \text{B}f;$$

i.e., the projection of the C. G. of three equal masses at P, Q, R on BC coincides with the projection of the C. G. of the triangle ABC on BC, and, since the same thing follows for the projections on CA and AB, the two centres of gravity must themselves coincide.

7784. (By B. REYNOLDS, M.A.)—From the vertex A of the triangle ABC, perpendiculars are drawn to AB and AC, meeting the circum-circle in D and E. Show that the quadrilateral of ADBE (or ADCE) is equal in area to the triangle.

Solution by G. G. MORRICE, B.A.; R. KNOWLES, B.A., L.C.P.; *and others.*

Since EC and BD are diameters, $\angle$ EBC = DCB = a right angle; hence EB is parallel to DC; and if a straight line HAK be drawn through A, parallel to BC, cutting BE in H and CD in K (both at right angles), the sum of the areas AEB, ADC = $\frac{1}{2}$EB . AH + $\frac{1}{2}$CD . AK = $\frac{1}{2}$EB . BC (since EB = CD and HK = BC) = area of EBC; therefore AECD = AEBCD − EBC = AEBCD − AEB − ADC = ABC.



[If from A, B, C we draw perpendiculars to each of the sides (9 in all), then 3 of these will meet at the orthocentre O, and the other 6, in pairs, on the circum-circle, at D, E, F; and, since ΔAEB = AOB, and so on, all round, the whole hexagon AEBFCD = 2ΔABC; but BD, CE, AF, being all diameters intersecting at the centre G of the circle, it is clear also that ΔAGD = ΔBGF, and so on, all round; hence the quadrilateral AEBD (or AECD) = half the hexagon = ΔABC.]

4569. (By Professor SYLVESTER, F.R.S.)—If any unicursal cubic be given, and an arbitrary conic, having its asymptotes parallel to two of those of the cubic, be drawn through its double point, and from this point

rays be drawn to meet again the conic and the cubic, and if in any ray the intercepted segments be called ρ and σ, and in that ray a length R be measured from the double point such that $R = \rho + \lambda\sigma$, where λ is any arbitrary constant: prove that the locus of the extremity of R will be the most general cubic which can be drawn so as to have a node at the given point, subject to the condition that its three asymptotes are parallel respectively to the asymptotes of the given unicursal cubic.

Solution by W. J. C. SHARP, M.A.

Let $xy(x+my)-(ax^2+2bxy+cy^2) = 0$ be the equation to the given cubic, referred to axes parallel to two of the asymptotes through the double point, and let $x = \sigma\mu$, $y = \sigma\nu$, therefore $\sigma = \dfrac{a\mu^2+2b\mu\nu+c\nu^2}{\mu\nu(\mu+m\nu)}$;
and let $xy-2(fy+gx) = 0$ be the conic, therefore

$$\rho = \frac{2(g\mu+f\nu)}{\mu\nu}, \text{ therefore } R = \rho+\lambda\sigma = \frac{2(g\mu+f\nu)}{\mu\nu}+\frac{\lambda(a\mu^2+2b\mu\nu+c\nu^2)}{\mu\nu(\mu+m\nu)},$$

therefore $R^3\mu\nu(\mu+m\nu) = 2(g\mu+f\nu)(\mu+m\nu)R^2+\lambda(a\mu^2+2b\mu\nu+c\nu^2)R^2$,

or $\quad xy(x+my) = (2g+a\lambda)x^2+(2f+2gm+2b\lambda)xy+(2mf+c\lambda)y^2$,

is the equation to the locus, which may be identified with any cubic fulfilling the conditions $xy(x+my)-(Ax^2+2Bxy+Cy^2) = 0$, by solving the equations $2g+a\lambda = A$, $f+gm+b\lambda = B$, $2mf+c\lambda = C$, which determine f, g, and λ uniquely.

7026. (By Sir JAMES COCKLE, M.A., F.R.S.)—Find sets of values (for xample, $x, y, z = 3, 4, 6$) which shall make each of the expressions

$$x^2+(x+1)y,\quad x^2+(x+1)(y+z),\quad x^2+(x+1)xz,\quad (x-1)(z-y),$$
$$(xy+z)^2-x(x-1)^2yz \text{ a rational square.}$$

Solution by the PROPOSER.

Let λ and μ be arbitrary; then

$$\begin{array}{lll} x=3, & y=\lambda(\lambda+3), & z=3\lambda(\lambda+1); \\ x=1, & y=\tfrac{1}{4}(\lambda^2-1), & z=\tfrac{1}{4}\left(\dfrac{1}{4\lambda^2}-1\right); \\ x=-1, & y=\tfrac{1}{2}(\lambda^2+\mu^2), & z=\pm\lambda\mu; \\ x=-3, & y=4, & z=0; \end{array}$$

are systems of solutions, for the first of which the squares are

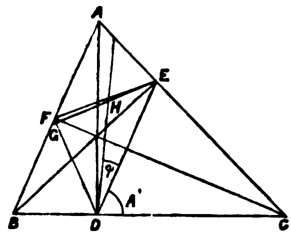
$$(2\lambda+3)^2,\ (4\lambda+3)^2,\ (6\lambda+3)^2,\ (2\lambda)^2,\ (6\lambda)^2.$$

When $\lambda^2+\mu^2$ is a square, another system is

$$x=0,\quad y=\lambda^2+\mu^2,\quad z=\pm 2\lambda\mu.$$

The first set gives $x^2+(x+1)(z-2y) = (2\lambda-3)^2$.

GRAPHICAL CONSTRUCTION (1) FOR CUBING A NUMBER, BY R. TUCKER, M.A., WITH (2) NOTE THEREON, BY PROFESSOR J. NEUBERG.



1. DEF is the pedal triangle, EG perpendicular on DF, EH = HG,

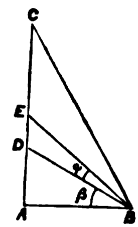
$$\angle \mathrm{DEH} = 2\mathrm{A} - \tfrac{1}{2}\pi,$$

therefore $\angle \mathrm{DHG} = \phi + 2\mathrm{A} - \frac{1}{2}\pi$,
and from $\triangle$ DEH,

$$-\frac{\sin\phi}{\cos(2\mathrm{A}+\phi)} = \frac{\mathrm{EG}}{2\mathrm{DE}} = \frac{\sin 2\mathrm{A}}{2} = \frac{\sin[\mathrm{A}-(\mathrm{A}+\phi)]}{\cos[\mathrm{A}+(\mathrm{A}+\phi)]},$$

whence $\tan(\mathrm{A}+\phi) = \tan^3\mathrm{A}$.
This gives a graphical construction for cubing any number ($n = \tan \mathrm{A}$).

2. La formule (1) revient à ceci :—
Si BE est la médiane, BD la bissectrice dans le triangle rectangle ABC, on a : $\mathrm{tg}\,\phi = \mathrm{tg}^3\,\beta$.

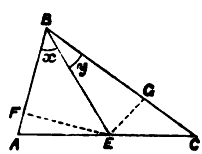


Considérons d'abord un triangle quelconque et ABC, AE = EC. Si l'on mène EF, EG perpendiculaires sur AB, AC, on a :

$$\frac{\mathrm{EF}}{\mathrm{EG}} = \frac{\mathrm{BE}\sin x}{\mathrm{BE}\sin y} = \frac{\mathrm{AE}\sin\mathrm{A}}{\mathrm{EC}\sin\mathrm{C}};$$

d'où
$$\frac{\sin x + \sin y}{\sin x - \sin y} = \frac{\sin\mathrm{A}+\sin\mathrm{C}}{\sin\mathrm{A}-\sin\mathrm{C}},$$

$$\frac{\mathrm{tg}\,\frac{1}{2}(x+y)}{\mathrm{tg}\,\frac{1}{2}(x-y)} = \frac{\mathrm{tg}\,\frac{1}{2}(\mathrm{A}+\mathrm{C})}{\mathrm{tg}\,\frac{1}{2}(\mathrm{A}-\mathrm{C})}.$$



Mais $\mathrm{tg}\,\frac{1}{2}(x+y) = \mathrm{tg}\,\frac{1}{2}\mathrm{B}$, $\mathrm{tg}\,\frac{1}{2}(\mathrm{A}+\mathrm{C}) = \cot\frac{1}{2}\mathrm{B}$;
donc $\mathrm{tg}\,\frac{1}{2}(x-y) = \mathrm{tg}^2\,\frac{1}{2}\mathrm{B}\,\mathrm{tg}\,\frac{1}{2}(\mathrm{A}-\mathrm{C})$.
Si $\mathrm{A} = \frac{1}{2}\pi$, on a : $\mathrm{tg}\,\phi = \mathrm{tg}^3\,\frac{1}{2}\mathrm{B}$.

4481. (By Professor SYLVESTER, F.R.S.)—Show how to obtain from its equation those points in a general cubic curve at which the angles between the four tangents drawn from it to other points of the curve taken two and two together are equal, and prove that the number of such points is in general 18.

Solution by W. J. C. SHARP, M.A.

If a line ($y = mx$) be drawn from the origin (a point on the curve) to cut the cubic, whose equation in Cartesian rectangular coordinates is

$$ax^3 + 3bx^2y + 3cxy^2 + dy^3 + 3ex^2 + 6fxy + 3gy^2 + 3hx + 3ky = 0,$$

the equation $(a + 3bm + 3cm^2 + dm^3)\,x^3 + 3\,(e + 2fm + gm^2)\,x^2 + 3\,(h + km)\,x = 0$

determines the points of intersection, and, if two of these coincide (*i.e.*, if the line be a tangent),

$$4(h+km)(a+3bm+3cm^2+dm^3)-3(e+2fm+gm^2)^2=0,$$

and the equation to the tangents from the origin is

$$(4dk-3g^2)y^4+4(3ck+dh-3fg)xy^3+6(2bk+2ch-2f^2-eg)x^2y^2$$
$$+4(ak+3bh-3ef)x^3y+(4ah-e^2)x^4=0 \quad\text{..............(1).}$$

Now, if the tangents are inclined as required, the equations to these referred to the bisectors of the angles between corresponding tangents (*i.e.*, to a system of rectangular coordinates) are $y+\lambda x=0$, $y-\lambda x=0$, $y+\mu x=0$, $y-\mu x=0$, and (1) must be reducible by an orthogonal transformation to the form $A\eta^4+6C\eta^2\xi^2+E\xi^4=0$. Now, writing (1), $(\alpha,\beta,\gamma,\delta,\epsilon\between y,x)^4=0$, and putting $y=\lambda\xi+\mu\eta$, $x=\mu\xi+\lambda\eta$, where $\lambda^2+\mu^2=1$, the reduction will be possible if

$$\beta\mu^4+(\alpha-3\gamma)\mu^3\lambda+3(\delta-\beta)\lambda^2\mu^2-(\epsilon-3\gamma)\lambda^3\mu-\delta\lambda^4=0,$$

and $$-\beta\lambda^4+(\alpha-3\gamma)\lambda^3\mu+3(\beta-\delta)\lambda^2\mu^2-(\epsilon-3\gamma)\lambda\mu^3+\delta\mu^4=0,$$

from which and the equation $\lambda^2+\mu^2=1$ it follows that

$$(\alpha-\epsilon)[3\gamma(\beta+\delta)-2\alpha\delta-2\beta\epsilon]-4(\beta-\delta)(\beta+\delta)^2=0 \quad\text{............(2),}$$

the condition that the tangents from the origin should be inclined as required.

If $Ax^3+3Bx^2y+3Cxy^2+Dy^3+3(Ex^2+2Fxy+Gy^2)+3(Hx+Ky)+L=0$ be the rectangular Cartesian equation to any cubic, and if this becomes

$$ax^3+3bx^2y+\ldots+3(hx+ky)=0,$$

where the origin is changed to a point (ξ,η) on the curve, then

$$A\xi^3+3B\xi^2\eta+3C\xi\eta^2+D\eta^3+3(E\xi^2+2F\xi\eta+G\eta^2)+3(H\xi+K\eta)+L=0,$$

and a, b, c, d are equal to A, B, C, and D respectively, and independent of ξ and η, while e, f, and g are linear functions of those variables, and h and k quadratic functions; hence, when these values of the coefficients $\alpha, \beta, \gamma, \delta, \epsilon$ of (1) are substituted in (2), each will be of the second order in ξ and η, and the equation (2) will become that of a sextic locus on which the required points must lie—these points being the intersections of this locus and the cubic, and therefore in general 18 in number.

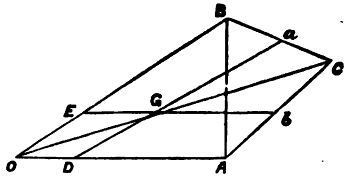
7782. (By W. J. C. SHARP, M.A.)—If the lines joining any point to the vertices of a triangle be similarly divided, prove that the lines joining the points of division to the mid-points of the corresponding sides are concurrent. If the lines joining any point to the vertices of a tetrahedron be similarly divided, prove that the lines joining the points of division to the centroids of the corresponding faces are concurrent.

Solutions by (1) W. E. HEAL, M.A.; (2) Rev. D. THOMAS, M.A.

1. Let a, b, c be the mid-points of the sides of the triangle ABC, and let OA, OB, OC be divided at D, E, F in the ratio of $1:n$; then the centre of gravity of W, W, W, nW, placed at A, B, C, O, will divide each of the lines Da, Eb, Fc in the ratio of $2:n+1$, and, since there can be only one centre of gravity, Da, Eb, Fc are concurrent.

In the case of a tetrahedron, suppose equal weights W placed at the angular points and a weight nW at the point O. The centre of gravity of the system will divide the given lines in the ratio of

$$n+1 : 3.$$



2. *Otherwise*:—If $\alpha, \beta, \gamma, \delta$ be the vectors of the vertices of the tetrahedron ABCD measured from the point O, the vector of the centroid of BCD will be $\frac{1}{3}(\beta+\gamma+\delta)$, and the line joining it to the point on OA will be $\rho = m\alpha(1-x)+\frac{1}{3}x(\beta+\gamma+\delta)$, and the other lines will be $\rho = m\beta(1-y)+\frac{1}{3}y(\gamma+\delta+\alpha)$, &c. At the point of intersection of these lines,

$$x = y = \frac{3m}{1+3m}, \text{ and } \rho = \frac{m}{1+3m}(\alpha+\beta+\gamma+\delta).$$

The symmetry of this result shows that the four lines are concurrent.

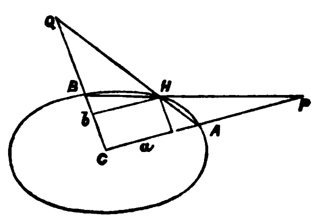
By a similar method the result for the triangle can be obtained, but in this case the three lines are evidently those joining the corresponding vertices of copolar triangles.

7812. (By Professor Genese, M.A.)—If CA, CB are semi-conjugate diameters of an ellipse, and P, Q two points on CA, CB produced such that AP . BQ = 2CA . CB, prove that BP, AQ intersect on the ellipse.

Solution by W. G. Lax, B.A.; R. Knowles, B.A.; *and others.*

Let CA, CB be taken as oblique axes of coordinates, and let coordinates of H be x, y, where AQ, BP intersect in H, so that

$$BQ . AP = 2 . BC . AC.$$



Let AC, BC = a, b.

Now $\dfrac{a-x}{y} = \dfrac{a}{CQ} = \dfrac{a}{b+BQ}$;

and $\dfrac{b-y}{x} = \dfrac{b}{CP} = \dfrac{b}{a+AP}$;

hence we have

$$\left.\begin{aligned}\frac{BQ}{a} &= \frac{y}{a-x}-\frac{b}{a}\\ \frac{AP}{b} &= \frac{x}{b-y}-\frac{a}{b}\end{aligned}\right\}, \therefore \frac{AP . BQ}{ab} = \frac{xy}{(a-x)(b-y)}+1-\frac{b}{a}\cdot\frac{x}{b-y}-\frac{a}{b}\cdot\frac{y}{a-x},$$

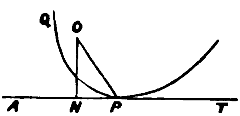
therefore $$2 = \frac{xy}{(a-x)(b-y)}+1-\frac{bx}{a(b-y)}-\frac{ay}{b(a-x)},$$

whence $$ab(a-x)(b-y) = abxy - bx . b(a-x) - ay . a(b-y),$$

from which $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, which is the equation to ellipse referred to conjugate diameters. Therefore H is on the ellipse of which AC, BC are conjugate diameters.

4865, 6880, 7212. (By the Editor.)—Find (1) a general expression for the locus of a point O in the plane of a curve that rolls on a given straight line, and apply it to the cases of (2) a parabola with O as focus, (3) a circle with O on the circumference, (4) a rectangular hyperbola, (5) a lemniscate, (6) a cardioid, (7) the curve $r^m = a^m \cos m\theta$; also show (8) that if s_1 be the length of a loop of the O-locus in (7), and s_2 the length of the loop of the original curve, then $s_1 s_2 = 2\left(\frac{1}{m}+1\right)\pi a^2$.

Solution by Âsûtosh Mukhopâdhyây ; N. Sarkar, M.A. ; *and others.*



1. Let $r^2 = \phi(p)$ be the p and r equation of any plane curve QP, referred to any point O in the plane of the curve as pole; then, if the curve rolls on the given line AT, the locus of the pole is easily found as follows:—Let Q be the point on the curve which was initially coincident with A. Draw ON perpendicular on AT, and join OP. Let AN $= x$, ON $= y$, OP $= r$, S = arc of the locus of O. Then, regarding P as the instantaneous centre, it is easy to see that the tangent to the curve-locus at O is at right angles to OP; hence

$$\cos \text{PON} = \frac{dx}{ds} = \frac{y}{r}, \text{ therefore } r^2 = y^2\left(\frac{ds}{dx}\right)^2 = y^2\left\{1+\left(\frac{dy}{dx}\right)^2\right\}.$$

But $p = \text{ON} = y$, therefore $\phi(y) = y^2\left\{1+\left(\frac{dy}{dx}\right)^2\right\}$, which is the differential equation of the required locus.

2. Here, $r = \frac{p^2}{a}$, therefore $\frac{dx}{ds} = \frac{a}{y}$, which is the differential equation to the catenary. [Otherwise proved in Vol. 25, p. 93.]

3. Here $r^2 = 2ap$, therefore $\frac{dx}{ds} = \left(\frac{y}{2a}\right)^{\frac{1}{2}}$, which is the differential equation to the cycloid.

4. Here $pr = a^2$, therefore $\frac{dx}{ds} = \frac{y^2}{a^2}$; and, since

$$\frac{dx}{ds} = \sin\phi, \quad y = \int ds \cos\phi,$$

the intrinsic equation to the locus is $\frac{ds}{d\phi} = \frac{a}{2}\,\frac{1}{(\sin\phi)^{\frac{1}{2}}}$.

5. Here $r^3 = a^2 p$, therefore $\frac{dx}{ds} = \left(\frac{y}{a}\right)^{\frac{2}{3}}$; hence the intrinsic equation to the locus is $\qquad \frac{ds}{d\phi} = \frac{3a}{2}(\sin\phi)^{\frac{1}{2}}$.

6. In the cardioid, which is the inverse of a parabola with respect to its focus, we have $r^3 = 2ap^2$, therefore $\frac{dx}{ds} = \left(\frac{y}{2a}\right)^{\frac{1}{3}}$, and the intrinsic equation to the locus is $\frac{ds}{d\phi} = 6a\sin^2\phi$, or $s = 3a\,(\phi - \sin\phi\cos\phi)$, which is, for its finite part, a curve of the cycloidal kind.

7. In the curve $r^m = a^m \cos m\theta$, we have $r^{m+1} = a^m p$, therefore

$$\frac{dx}{ds} = \left(\frac{y}{a}\right)^{\frac{m}{m+1}},$$

and the intrinsic equation to the locus of the pole is

$$\frac{ds}{d\phi} = a\left(1+\frac{1}{m}\right)(\sin\phi)^{\frac{1}{m}}.$$

8. The length of a loop of the curve in (7) is easily found to be

$$s_1 = a\left(1+\frac{1}{m}\right)\pi^{\frac{1}{2}}\,.\,\Gamma\left(\frac{1}{2}+\frac{1}{2m}\right) + \Gamma\left(1+\frac{1}{2m}\right).$$

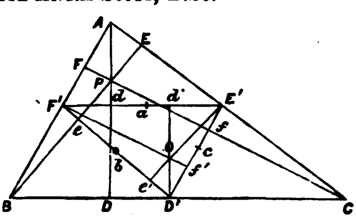
But, if s_2 be the length of the loop of the original curve, we have

$$s_2 = \frac{a}{m}\,\pi^{\frac{1}{2}}\,.\,\Gamma\left(\frac{1}{2m}\right) + \Gamma\left(\frac{1}{2}+\frac{1}{2m}\right).$$

Hence it follows that $s_1 s_2 = 2a^2\pi\left(1+\frac{1}{m}\right)$.

7900. (By R. Tucker, M.A.)—Prove that the diameter of the Brocard and Triplicate-Ratio circle which passes through the circum-centre, passes also through the orthocentre of the pedal triangle.

Solution by Charlotte Angas Scott, B.Sc.



Let O be circumcentre; P orthocentre; K Symmedian point; H N.P.C.-centre; a, b, c mid-points of E'F', &c.; G, G' the centroids of DEF, D'E'F'. G' is centre of gravity of three equal particles at D'E'F'. When these have translations D'D, E'E, F'F, G' becomes G, therefore G'G is parallel to translations compounded of D'D, E'E, F'F; *i.e.*, if we have forces acting at a point parallel and proportional to $d'd$, $e'e$, $f'f$, their resultant is parallel to G'G.

Now $d'a$ is equivalent to D'a and d'D', G'G is therefore parallel to resultant of six forces represented by D'a, E'b, F'c, and d'D', e'E', f'F', acting at a point. Take them to act at O. D'a, E'b, F'c are themselves in equilibrium. The remaining three are completely represented by the six dD', D'D, eE', E'E, fF', F'F; *i.e.*, resultant of the two groups (dD', eE', fF'), (D'D, E'E, F'F) passes through O, and is parallel to GG'. But we have shown that the resultant of the group (D'D, E'E, F'F) passes through

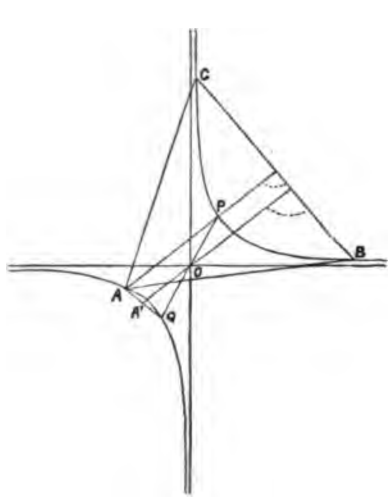
O and is parallel to GG′; therefore the resultant of the group (dD', eE', fF') must also pass through O and be parallel to GG′.

Now dD', eE', fF' meet in K; therefore the resultant must pass through K, *i.e.*, OK is parallel to GG′.

Now OK is the specified diameter; and GG′ is parallel to the line joining the orthocentres of DEF, D′E′F′; the orthocentre of D′E′F′ being O, this line must therefore be OK—*i.e.*, the specified diameter passes through the orthocentre of the pedal triangle.

7816. (By Asparagus.)—PQ is a diameter of a rectangular hyperbola, and a circle with centre P and radius PQ meets the hyperbola again in ABC; prove that ABC will be an equilateral triangle.

Solutions by Prof. Wolstenholme, Sc.D.; W. T. Mitchell, M.A.; *and others.*



1. Bisect QA in A′ and join OA′; then AQ, BC, being common chords of a circle and hyperbola, are equally inclined to the axes, and OA′ is the diameter conjugate to chords parallel to AQ, hence OA′ will be at right angles to BC. But AP is parallel to A′O; hence AP is at right angles to BC, and similarly BP is at right angles to CA, and P is the orthocentre of the triangle ABC. But P is also the circumcentre of the triangle ABC. Hence the triangle must be equilateral.

2. *Otherwise*:—Let $xy = c^2$ be the equation of the hyperbola, (cm, cm^{-1}) the point P, then the equation of the circle will be

$$(x-cm)^2+(y-cm^{-1})^2 = 4c^2(m^2+m^{-2}).$$

Let $(c\mu, c\mu^{-1})$ be a point where this circle meets the hyperbola again, then

$$(\mu-m)^2-(\mu^{-1}-m^{-1})^2 = 4m^2+4m^{-2},$$

or
$$(\mu+m)(\mu-3m)+(\mu^{-1}+m^{-1})(\mu^{-1}-3m^{-1}) = 0;$$

or, rejectingthe fact or $\mu+m$, which gives the point Q, we get for the points A, B, C, the cubic in μ,

$$\mu^3-3m\mu^2-3\frac{\mu}{m}+\frac{1}{m} = 0,$$

whence, if m_1, m_2, m_3 be the three roots,

$$m_1+m_2+m_3 = 3m,\quad m_1^{-1}+m_2^{-1}+m_3^{-1} = 3m^{-1},$$

which equations prove that the centroid of the triangle ABC is the point P; or the centroid coincides with the circum-centre, and the triangle is equilateral.

[Or, we have also the equation $m_1m_2m_3 = -m^{-1}$, which proves that P is the orthocentre of the triangle ABC, so that the orthocentre and the circum-centre coincide, and the triangle is therefore equilateral.]

Hence, the three equations

$$m_1+m_2+m_3 = 3m,\quad m_1^{-1}+m_2^{-1}+m_3^{-1} = 3m^{-1},\quad mm_1m_2m_3 = -1,$$

must be equivalent to the three,

$$(m_2-m_3)^2\left(1+\frac{1}{m_2^2m_3^2}\right) = 12\left(m^2+\frac{1}{m^2}\right),$$

and the two similar equations; the proof of which is a rather nice algebraical exercise.

7813. (By Professor Cochez.)—Trouver une courbe telle que l'arc compté à partir d'un point fixe soit moyenne proportionnelle entre l'ordonnée et le double de l'abscisse.

Solution by A. Gordon, M.A.; W. T. Mitchell, M.A.; *and others.*

The condition gives $s-l = \sqrt{2xy}$: put $p = \dfrac{dy}{dx}$, therefore

$$p^2(2xy-x^2)-2pxy = y^2-2xy,\ \text{therefore}\ p = \frac{y}{2y-x}\pm\frac{\sqrt{2y}\,(y-x)}{(2y-x)\sqrt{x}}.$$

By the substitutions $y = vx$, $\sqrt{2v}+1 = \xi = \left(1+\sqrt{\dfrac{2y}{x}}\right)$,

$$\tfrac{1}{2}\frac{dx}{x} = \frac{\xi\,d\xi}{2-(\xi-1)^2},\ \text{therefore}\ -\tfrac{1}{2}\log(x-y)+\frac{1}{2\sqrt{2}}\log\frac{\sqrt{y}+\sqrt{x}}{\sqrt{y}-\sqrt{x}} = \text{const.},$$

or $$c = (x-y)^{\sqrt{2}}\,(\sqrt{y}-\sqrt{x})^2.$$

7716 & 7961. (By J. J. Walker, M.A., F.R.S.)—Find the conditions that, in the working of the suction pump, the water shall rise in the suction tube in the second stroke higher than, just as high as, or not so high as, it rose in the first stroke.

Solution by the Proposer.

Suppose H to be the height of the water barometer in centimetres, b the height of the bottom of the working barrel above the surface of the water

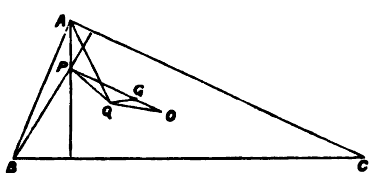
in the well; A, a the sections of the barrel and suction tube in sq. cms.; x the length of first stroke which will raise the water h cms. in the tube, y the length of second stroke necessary to raise it h cms. higher; *i.e.*, to a total height of $2h$ cms., $2h$ being not greater than b. The following conditions hold: $[Ax + a(b-h)](H-h) = abH$,

$$[Ay + a(b-2h)](H-2h) = a(b-h)(H-h),$$

or (1) $Ax = ah[1 + b/(H-h)]$, $Ay = ah[1+(b-h)/(H-2h)]$ (2).
Hence y will be $> = $ or $< x$, as $(b-h)(h-H)$ is $> =$ or $< b(H-2h)$, viz., as h is $> =$ or $< H-b$; or, what is the same thing, the two strokes of the piston being, as usual, of the same length (a), the rise of water in the suction tube produced by the second will be $< =$ or $>$ that resulting from the first stroke as h is $> =$ or $< H-b$. Put $h = H-b+k$, then, from (1), $b[Aa-2a(H-b)] = k[Aa + a(2b-k)]$, so that k must be of the same affection as $Aa-2a(H-b)$, since, if positive, it must be $< b$.

3733. (By R. Tucker, M.A.)—Triangles are inscribed in a circle (O), P is the orthocentre, and Q the inscribed centre; prove that the area of the triangle OPQ varies as $\sin\frac{1}{2}(A-B)\sin\frac{1}{2}(B-C)\sin\frac{1}{2}(C-A)$.

Solution by Rev. T. C. Simmons, M.A.



On OP take $OG = \frac{1}{3}OP$; then G is the centre of mean position of the points A, B, C; and we have

$$\Delta OPQ = \tfrac{3}{2}\Delta GPQ$$
$$= \tfrac{1}{2}(\Delta APQ + \Delta BPQ + \Delta CPQ),$$

each triangle being considered positive or negative according as it lies on one side or the other of PQ.

Now, $\angle PAQ = \angle BAQ - \angle BAP = \frac{1}{2}A - (90^\circ - B) = \frac{1}{2}(B-C)$;

$$AP = c\cos A \operatorname{cosec} C = 2R\cos A, \quad AQ = r\operatorname{cosec}\tfrac{1}{2}A.$$

Hence $\frac{1}{2}\Delta APQ = \frac{1}{4}AP \,.\, AQ\sin PAQ = \frac{1}{2}Rr\cos A\operatorname{cosec}\frac{1}{2}A\sin\frac{1}{2}(B-C)$

$$= 2R^2\cos A\sin\tfrac{1}{2}B\sin\tfrac{1}{2}C\sin\tfrac{1}{2}(B-C)$$
$$= 2R^2[\cos A\sin^2\tfrac{1}{2}B\sin\tfrac{1}{2}C\cos\tfrac{1}{2}C - \cos A\sin^2\tfrac{1}{2}C\sin\tfrac{1}{2}B\cos\tfrac{1}{2}B]$$
$$= R^2[\sin^2\tfrac{1}{2}B\sin C\cos A - \sin^2\tfrac{1}{2}C\sin B\cos A].$$

Adding this and two similar expressions, we obtain

$$R^2[\sin^2\tfrac{1}{2}B\sin(C-A) + \sin^2\tfrac{1}{2}C\sin(A-B) + \sin^2\tfrac{1}{2}A\sin(B-C)]$$
$$= \tfrac{1}{2}R^2[\sin(C-A) + \sin(A-B) + \sin(B-C) + \cos(C+A)\sin(C-A)$$
$$+ \cos(A+B)\sin(A-B) + \cos(B+C)\sin(B-C)]$$
$$= \tfrac{1}{2}R^2[\sin(C-A) + \sin(A-B) + \sin(B-C)]$$
$$= R^2[\sin\tfrac{1}{2}(C-A)\cos\tfrac{1}{2}(C-A) - \sin\tfrac{1}{2}(C-A)\cos\tfrac{1}{2}(C+A-2B)]$$
$$= -2R^2\sin\tfrac{1}{2}(A-B)\sin\tfrac{1}{2}(B-C)\sin\tfrac{1}{2}(C-A),$$

the expression for the area of OPQ.

7845. (By the FATHER OF THE FIFTEEN YOUNG LADIES.)—

From the Lancashire Witches, the direst alive,
The most dangerous twelve of them all
Are bidden in sixes, repeating no five,
For a year, to the Monthly Ball.

Fear leaves the arrangement to them; so they use
The lot, far better than fighting,
To settle the turn of each beauty to choose
Her party, and do the inviting:
Provided that all, or there would have been fights,
Shall dazzle and kill on the first two nights;

And, as odd's ill in witchery, every one
Shall appear with another times even or none.

K's turn is the first; and provident K
From every one, B, of her train,
Insists on a promise, that B on her day
Shall choose her good K back again:
And every month the enchanting inviter
Requires of her bevy thus all to requite her.

Now, prove by a dozen of sextuplets
That, no matter who the first turn gets
And no matter how the turns of the sets
We alter, the chosen will pay their debts.

Solution by the PROPOSER.

This question is a riddle which has been found worthy of the steel of bright and sharp bodkins.

There are only two solutions. Let A and *a*, B and *b*, &c. be opposite faces of the regular 12-edron. For one solution, write A with its collaterals, *a* with its collaterals, &c. For the second, write A with the collaterals of *a*, *a* with the collaterals of A, &c.

7828. (By ASÛTOSH MUKHOPÂDHYÂY.)—Prove that the integral of

$$\frac{d^2y}{dx^2} - c^2x^{-\frac{8}{5}}\,y = 0$$

is $$y = \left(x^{\frac{2}{5}} - \frac{3}{5c}x^{\frac{1}{5}} + \frac{3}{25c^2}\right) A\epsilon^{5cx^{\frac{1}{5}}} + \left(x^{\frac{2}{5}} + \frac{3}{5c}x^{\frac{1}{5}} + \frac{3}{25c^2}\right) B\epsilon^{-5cx^{\frac{1}{5}}}.$$

[In GREGORY'S *Examples* (1846), p. 345, the integral is given to be

$$y = \left(x^{\frac{2}{5}} - \frac{3}{5c}x^{\frac{1}{5}}\right) A\epsilon^{5cx^{\frac{1}{5}}} + \frac{3}{5c}\left(x^{\frac{1}{5}} + \frac{3}{5c}\right) B\epsilon^{-5cx^{\frac{1}{5}}}.\;]$$

Solutions by (1) ROBERT RAWSON, F.R.A.S.; (2) Prof. WILLIAMSON, F.R.S.

(1) Assume $$\mathrm{P}y = \int_{-h}^{h} \epsilon^{\mathrm{Q}t}\,(h^2 - t^2)^n\,dt \quad\ldots\ldots\ldots\ldots(1),$$

where P, Q are given functions of x; h, n constant quantities. The definite integral (1) can be evaluated when (n) is a positive integer. Differentiate (1) with respect to (x), and the result with respect to x, then

$$\frac{d\mathrm{P}y}{dx} \div \frac{d\mathrm{Q}}{dx} = \int_{-h}^{h} \epsilon^{\mathrm{Q}t}\,(h^2 - t^2)^n\,t\,dt \quad\ldots\ldots\ldots\ldots(2),$$

$$\frac{d}{dx}\left\{\frac{d\mathrm{P}y}{dx} \div \frac{d\mathrm{Q}}{dx}\right\} = \frac{d\mathrm{Q}}{dx}\int_{-h}^{h} \epsilon^{\mathrm{Q}t}\,(h^2 - t^2)^n\,t^2\,dt \quad\ldots\ldots\ldots\ldots(3),$$

(1) $\times h^2 \dfrac{dQ}{dx}$ − (3) gives

$$h^2 P \frac{dQ}{dx} y - \frac{d}{dx}\left\{\frac{dPy}{dx} + \frac{dQ}{dx}\right\} = \frac{dQ}{dx}\int_{-h}^{h} \epsilon^{Qt}(h^2-t^2)^{n+1}\,dt \;\ldots\ldots\ldots(4).$$

Integrate (2) by parts, then

$$\frac{(2n+2)\dfrac{dPy}{dx}}{Q} = \frac{dQ}{dx}\int_{-h}^{h} \epsilon^{Qt}(h^2-t^2)^{n+1}\,dt = h^2 P \frac{dQ}{dx} y - \frac{d}{dx}\left\{\frac{dPy}{dx} + \frac{dQ}{dx}\right\},$$

from (4), which reduces to

$$y'' + \left\{\frac{2P'}{P} + (2n+2)\frac{Q'}{Q} - \frac{Q''}{Q'}\right\} y' + \left\{\frac{P''}{P} + (2n+2)\frac{P'}{P}\frac{Q'}{Q} - \frac{P'}{P}\frac{Q''}{Q'} - h^2 (Q')^2\right\} y$$
$$= 0 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5).$$

The evaluation of the definite integral (1) depends, therefore, upon the solution of (5). And the differential equation (5) depends upon the evaluation of the definite integral (1). Equation (1) is only a particular solution of (5), but equation (5) is not altered by changing the sign of Q; hence another particular solution of (5) is

$$Py = \int_{-h}^{h} \epsilon^{-Qt}(h^2-t^2)^n\,dt \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6).$$

Several important and historical differential equations are included in (5), whose general solution is represented by (1) and (6).

If $P = \dfrac{1}{x}$, $Q = 5cx$, $h = 1$, $n = 2$, then (5) becomes

$$\frac{d^2y}{dx^2} - c^2x^{-\frac{8}{5}}\,y = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7),$$

which is the equation in the question, and whose general solution is the sum of the two particular solutions (1) and (6), each multiplied by an arbitrary constant; then, by integration,

$$y = x\int_{-1}^{1} \epsilon^{5cx^{\frac{1}{5}}t}(1-t^2)^2\,dt$$

$$= \frac{A+B}{2}\left(x^{\frac{2}{5}} - \frac{3x^{\frac{1}{5}}}{5c} + \frac{3}{25c^2}\right)\epsilon^{5cx^{\frac{1}{5}}} + \frac{A+B}{2}\left(x^{\frac{2}{5}} + \frac{3x^{\frac{1}{5}}}{5c} + \frac{3}{25c^3}\right)\epsilon^{-5cx^{\frac{1}{5}}}, \text{ from (1)},$$

and

$$y = x\int_{-1}^{1} \epsilon^{-5cx^{\frac{1}{5}}t}(1-t^2)^2\,dt$$

$$= -\frac{A-B}{2}\left(x^{\frac{2}{5}} + \frac{3x^{\frac{1}{5}}}{5c} + \frac{3}{25c^2}\right)\epsilon^{-5cx^{\frac{1}{5}}} + \frac{A-B}{2}\left(x^{\frac{2}{5}} - \frac{3x^{\frac{1}{5}}}{5c} + \frac{3}{25c^3}\right)\epsilon^{5cx^{\frac{1}{5}}}, \text{ from}$$

(6). The sum of these integrals gives the general integral in the question, which is quite correct, and the integral in GREGORY's *Examples* is in error. It may be stated that the latter part of the question in GREGORY's *Examples* is correct when (r) is a positive integer.

If $P = x^p$, $Q = ax^r$, then (5) becomes

$$y'' + \frac{2p + (2n+1)r + 1}{x}\cdot y' + \left\{\frac{p[p+(2n+1)]r}{x^2} - h^2a^2r^2x^{2r-2}\right\} y = 0 \ldots(8),$$

whose general integral is

$$x^p y = \int_{-h}^{h} \left\{ c\epsilon^{ax^r t} + c_1 \epsilon^{-ax^r t} \right\} (h^2 - t^2)^n \, dt \quad \text{..............(9)},$$

which is always integrable when (n) is a positive integer. In equations (8) and (9), put, for (p) and (r), $p = 0$ and $(2n+1)r + 1 = 0$, then

$$y = \int_{-h}^{h} \left\{ c\epsilon^{ax^r t} + c_1 \epsilon^{-ax^r t} \right\} (h^2 - t^2)^n \, dt \quad \text{..............(10)}$$

is the general solution of $y'' = h^2 a^2 r^2 x^{-\frac{4n+4}{2n+1}} . y$ ....(11).

In (8) and (9) substitute the values $p = -1$ and $(2n+1)r + 1 = 0$, then

$$y = x \int_{-h}^{h} \left\{ c\epsilon^{ax^r t} + c_1 \epsilon^{-ax^r t} \right\} (h^2 - t^2)^n \, dt \quad \text{..............(12)}$$

is the general solution of $y'' = h^2 a^2 r^2 x^{-\frac{4n}{2n+1}} . y$ ....(13).

Equations (11) and (13) are known as RICCATI's.

In (8) and (9), put $r = 1$ and $p = -n-1$, then

$$y = x^{n+1} \int_{-h}^{h} \left\{ c\epsilon^{axt} + c_1 \epsilon^{-axt} \right\} (h^2 - t^2)^n \, dt \quad \text{...........(14)}$$

is the general solution of GASKIN's differential equation, viz.,

$$\frac{d^2y}{dx^2} - h^2 a^2 y = \frac{n(n+1)}{x^2} y \text{..........................(15)}.$$

In (8) and (9) put $r = 1$, $a = \sqrt{-1}$, $h = 1$, and $2p + 2n + 1 = 0$, then

$$y = x^{n+\frac{1}{2}} \int_{-1}^{1} \left\{ c\epsilon^{xt\sqrt{-1}} + c_1 \epsilon^{-xt\sqrt{-1}} \right\} (1 - t^2)^n \, dt \quad \text{.........(16)}$$

is the general solution of $\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \left\{ 1 - \frac{(n+\frac{1}{2})^2}{x^2} \right\} y = 0$.........(17),

which is the well-known equation of BESSEL's functions.

In (8) and (9), put $p + 2n + 1 = 0$ and $2p + (2n+1)r + 1 = 2m$, from which $p = -2n-1$ and $r = \frac{2m+4n+1}{2n+1}$, then

$$y = x^{2n+1} \int_{-h}^{h} \left\{ c\epsilon^{ax^r t} + c_1 \epsilon^{-ax^r t} \right\} (h^2 - t^2)^n \, dt \quad \text{...........(18)}$$

is the general solution of $\frac{d^2y}{dx^2} + \frac{2m}{x}\frac{dy}{dx} = h^2 a^2 r^2 x^{\frac{4(m+n)}{2n+1}} y$ ....(19).

See BOOLE's *Differential Equations*, p. 458, Ex. 4.

(2) To find the integral of $\frac{d^2y}{dx^2} - c^2 x^{-\frac{8}{5}} y = 0$, multiply by x^2, then the expression is readily transformed into

$$\left\{ x\frac{d}{dx}\left(x\frac{d}{dx} - 1 \right) - c^2 x^{\frac{2}{5}} \right\} y = 0.$$

Let $x = z^5$, and it is easily seen that $x\frac{d}{dx} \equiv \frac{z}{5}\frac{d}{dz}$.

Hence we get $\quad [zD(zD-5)-a^2z^2]\,y = 0,$

where D stands for $\dfrac{d}{dz}$, and $a = 5c$.

Now, by a transformation analogous to that of Boole [*Differential Equations*, p. 418], we assume $y = (zD-1)(zD-3)\,u$,
and the transformed equation is readily seen to give

$$[zD(zD-1)-a^2z^2]\,u = 0, \quad \text{hence } u = Ae^{az}+Be^{-az}.$$

Therefore $\quad y = (zD-1)(zD-3)\,u = (z^2D^2-3zD+3)\,u$

$$= Ae^{az}\,[a^2z^2-3az+3]+Be^{-az}\,[a^2z^2+3az+3].$$

This agrees with the answer given by the Proposer. I may add that the method can be readily applied to Riccati's equation when written in the form $\quad \dfrac{d^2y}{dx^2}+ax^my = 0.$

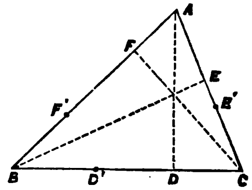
[The solution of all differential equations of the form

$$\left\{\frac{d^2}{dx^2}\pm\left(\frac{c}{2n\pm1}\right)^2 x^{-\frac{4n}{2n\pm1}}+\frac{(n\,.\,n\pm1-m\,.\,m+1)\,x^{-2}}{(2n\pm1)^2}\right\}y = 0,$$

with special reference to the particular case here mentioned—the incompleteness of Gregory's solution,—and the deduction of the true integral, will be found fully developed in Dr. Curtis' paper, published in the *Cambridge and Dublin Mathematical Journal* for November, 1854.]

7819. (By R. Tucker, M.A.)—AD, BE, CF are the perpendiculars from the angles on the sides of ABC: BD′ = CD, CE′ = AE, BF′ = AF are taken on the same sides; prove that AD′, BE′, CF′ pass through a point (π), and that the triangle D′E′F′ = ΔDEF. Also, if perpendiculars to the sides through D′, E′, F′ intersect in π', then this point lies on the line through the centroid and circumcentre of ABC.

Solutions by (1) Rev. D. Thomas, M.A.; (2) Rev. T. C. Simmons, M.A.



If α, β, γ be respectively the vectors of A, B, C measured from a point O, the vectors α', β', γ' of D′, E′, F′ will be respectively

$$\frac{c\cos B\,\beta+b\cos C\,\gamma}{a},\quad \frac{a\cos C\,\gamma+c\cos A\,\alpha}{b},\quad \frac{b\cos A\,\alpha+a\cos B\,\beta}{c},$$

and ρ the vector of the point of intersection of AD′ and BE′

$$= (1-x)\,\alpha+\frac{x}{a}\,(c\beta\cos B+b\gamma\cos C) = (1-y)\,\beta+\frac{y}{b}\,(a\gamma\cos C+ca\cos A).$$

Hence $\quad x = \dfrac{2a^2}{a^2+b^2+c^2}$ and $\rho = 2\,(bc\cos A\,\alpha+ca\cos B\,\beta+ab\cos C\,\gamma)$,

and we see, from the symmetry of the value of ρ, that AD′, BE′, CF′ are concurrent. Also

$$2 \text{ area } D'E'F' = TV(\beta'\gamma' + \gamma'\alpha' + \alpha'\beta') = 2 \cos A \cos B \cos C \,.\, TV(\beta\gamma + \gamma\alpha + \alpha\beta)$$
$$= 4 \cos A \cos B \cos C \text{ area } ABC = 2 \text{ area } DEF,$$

therefore $$\text{area } D'E'F' = \text{area } DEF.$$

The vectors of O_c, π', and centroid are respectively

$$(4 \sin A \sin B \sin C)^{-1} [\alpha \sin 2A + \beta \sin 2B + \gamma \sin 2C] \quad \ldots\ldots\ldots(1),$$
$$(2 \sin A \sin B \sin C)^{-1} [(\sin 2A - 2 \sin A \cos B \cos C)\,\alpha$$
$$+ (\sin 2B - 2 \sin B \cos A \cos C)\,\beta + (\sin 2C - 2 \sin C \cos A \cos B)\,\gamma] \ldots(2)$$
$$\tfrac{1}{3}(\alpha + \beta + \gamma) \ldots\ldots\ldots\ldots\ldots\ldots\ldots(3);$$

and because $(1) - (2) = [(2) - (3)] \times$ scalar, O_c, π', and centroid are on the same right line.

2. *Otherwise* :—Since AD, BE, CF are concurrent, therefore

AE . CD . BF = EC . DB . FA ; that is, CE′. BD′. AF′ = AE′. CD′. BF′ ;

therefore AD′, BE′, CF′ are concurrent. Again, let

CD or BD′ = l . BC ; AE or CE′ = m . AC ; AF = BF′ = n . AB.

Then it can easily be shown that the triangles DEF, D′E′F′ are each to the triangle ABC in the ratio of $(1-l)(1-m)(1-n) + lmn$ to unity, whence $\Delta DEF = \Delta D'E'F'$.

Lastly, it is evident that π' will be the orthocentre of the triangle (L) formed by drawing through A, B, C parallels to BC, CA, AB respectively: and that the circumcentre of ABC will coincide with O, the orthocentre of the triangle (M) formed by joining the mid-points of AB, BC, CA. But G, the centroid of ABC, is the centre of similitude of (L) and (M); therefore π'GO is a straight line.

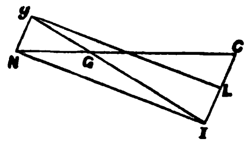
[From this second solution, it will be seen that the first two parts of the Question hold when AD, BE, CF are *any* three concurrent lines drawn from A, B, C to meet the opposite sides.]

7832. (By Rev. T. C. Simmons, M.A.)—In a plane triangle prove that the in-centre, the nine-point centre, the centroid of the perimeter, and the point midway between the in-centre and the circum-centre, lie at the four corners of a parallelogram.

Solution by Dr. Curtis ; B. Hanumanta Rau, M.A. ; *and others.*

If two similar triangles, M, N, the ratio of whose corresponding sides is $m : n$, be situated in a plane, α being the angle at which any corresponding pair of sides are inclined, and any two points, C, I, being assumed, two other points, c, i, be found, geometrically related to N in the same way as C, I are to M ; then obviously the line CI is inclined to ci at the angle α, and CI : ci :: $m : n$. As a particular case, if M be any triangle and N the triangle obtained by joining the middle points of the sides of M, $m : n :: 2 : 1$, $\alpha = \pi$, CI is parallel to ic, and CI = $2ic$; therefore, if CI be bisected in L, it follows that L, I, i, c are the corners of a parallelogram. This includes the case in which C, I are the circum-centre and in-centre of M, and c, i the circum-centre and in-centre of N, but the circle circumscrib-

ing N is the nine-point circle of M, while the centroid of the perimeter of the triangle M is the in-centre of the triangle N.



[Let C be the circum-centre, I the in-centre, N the mid-centre, G the centroid of the triangle, and g the centroid of its perimeter; then, since G is the centre of similitude of the original triangle and that formed by joining the mid-points of the sides, and g is the in-centre of the latter, therefore $GI = 2Gg$, also $GC = 2GN$, therefore $CI = 2Ng$, and is parallel to Ng; hence, if CI be bisected in L, LINg is a parallelogram.]

7827. (By B. HANUMANTA RAU, M.A.)—Show that the value of x from the equation $x^{x+1} = x+1$ is 1·4414 nearly.

Solution by D. BIDDLE.

$$x^{x+1} = x+1, \text{ therefore } (x+1)(\log x) = \log(x+1).$$

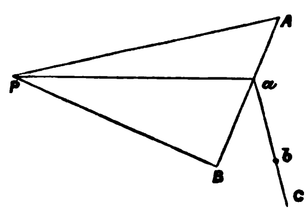
Now $\qquad \log_e(x+1) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \ldots,$

and $\qquad \log_e x = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \ldots$

By carrying each of these series to an indefinite number of terms and utilising the results in the given equation, we could with immense labour obtain the approximate value of x. But the Tables of Logarithms enable us readily to arrive step by step at the following approximate values:— $1 < x < 2$; $1·4 < x < 1·5$; $1·44 < x < 1·45$; $1·441 < x < 1·442$; and finally $x = 1·4414$ nearly.

7824. (By A. H. CURTIS, LL.D, D.Sc. Suggested by Quest. 7771.) —Given any number of points in space A, B, C, D, &c., find the locus of a point P which moves so that the length of the resultant of the translations lPA, mPB, nPC, pPD, &c. is constant, l, m, n, p, &c. being given numbers.

Solution by B. HANUMANTA RAU, M.A.; N. SARKAR, M.A.; *and others.*



Divide AB in a such that

$$Aa : aB = m : l.$$

Then step

$l \,.\, Pa = l\,(PA + Aa) = l \,.\, PA + l \,.\, Aa,$

and step $m \,.\, Pa = m \,.\, PB + m \,.\, Ba,$

but $\qquad l \,.\, Aa + m \,.\, BA = 0,$

therefore $(l+m)\,Pa = l \,.\, PA + m \,.\, PB.$

Again, take B in aC such that

$$ab : bC = n : m+l,$$

$$(l+m+n)\,Pb = (l+m)\,Pa + n \,.\, PC = l \,.\, PA + m \,.\, PB + n \,.\, PC.$$

Similarly for all the translations. Thus, if G is the centre of gravity of weights lW, mW, nW ... placed at A, B, C, D, then PG $(l+m+n+\ldots)$ = resultant of all the translations. The locus of P is therefore a circle with centre G.

7928. (By Professor Wolstenholme, M.A., Sc.D.)—Prove that the polar circle of a triangle ABC intersects the circum-circle and the nine-point circle, each at the angle $\cos^{-1}(-\cos A\cos B\cos C)^{\frac{1}{2}}$.

Solution by D. Edwardes; Rev. T. C. Simmons, M.A.; *and others.*

The radius of the polar circle is $2R(-\cos A\cos B\cos C)^{\frac{1}{2}}$, and the distance between the orthocentre and circumcentre is

$$R(1-8\cos A\cos B\cos C)^{\frac{1}{2}};$$

therefore
$$\cos\theta = \frac{1-8\cos A\cos B\cos C-1+4\cos A\cos B\cos C}{4(-\cos A\cos B\cos C)^{\frac{1}{2}}} = (-\cos A\cos B\cos C)^{\frac{1}{2}}.$$

The distance between the orthocentre and the nine-point centre is

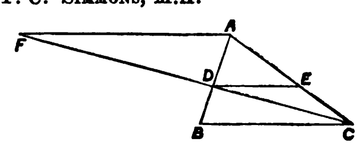
$$\tfrac{1}{2}R(1-8\cos A\cos B\cos C)^{\frac{1}{2}};$$

hence
$$\cos\phi = \frac{-4\cos A\cos B\cos C+\frac{1}{4}-\frac{1}{4}(1-8\cos A\cos B\cos C)}{2(-\cos A\cos B\cos C)^{\frac{1}{2}}} = (-\cos A\cos B\cos C)^{\frac{1}{2}}.$$

Since $\cos A\cos B\cos C$ is negative, we always obtain real values for θ and ϕ, so that the circles always intersect.

5706. (By the Editor.)—Parallel to the base BC of a triangle ABC draw a straight line DE, cutting the sides AB, AC in D, E, such that the squares on BD and CE shall be together equal to the square on DE.

Solution by Rev. T. C. Simmons, M.A.



Draw AF parallel to BC and of length equal to that of the hypotenuse of a right-angled triangle whose sides are AB, AC. Then FC will meet AB in the required point D. For, drawing DE parallel to BC, we have

$$DB^2 : AB^2 = CE^2 : CA^2 = DE^2 : AF^2,$$

$$\therefore\quad CE^2+DB^2 : DE^2 = CA^2+AB^2 : AF^2, \quad\text{or}\quad CE^2+DB^2 = DE^2.$$

Again, if FB be joined and produced to meet AC in E′, a second parallel D′E′ can be drawn satisfying the same condition.

[For the Proposer's solution of this Question, see Vol. xxxiii., p. 89.]

7724. (By B. H. RAU, M.A.)—Given two sides of a triangle in position, and the perimeter, prove that the locus of the mid-point of the third side is an hyperbola.

Solution by R. KNOWLES, B.A.; Professor MATZ, M.A.; *and others.*

Take AB, BC as axes, and put $Pm = y$, $Bm = x$, $\angle ABC = \omega$, perimeter $= 2c$; then $AB = 2y$, $BC = 2x$, and, from triangle ABC,

$$AC^2 = 4(x^2 + y^2 - 2xy\cos\omega) = 4(c - x - y)^2;$$

hence the locus is the hyperbola

$$2(1 + \cos\omega)\,xy - 2c(x + y) + c^2 = 0.$$

7913. (By ASÛTOSH MUKHOPÂDHYÂY.)—Tangents are drawn to a parabola, so that the intercepts they make on the directrix are in arithmetical progression; prove that the trigonometrical tangents of double the angles of inclination of the tangents to the directrix form a harmonic progression.

Solution by R. KNOWLES, B.A.; W. J. GREENSTREET, B.A.; *and others.*

The equation to the tangent at angle α to the axis is

$$x\cos^2\alpha + y\sin\alpha\cos\alpha + m = 0,$$

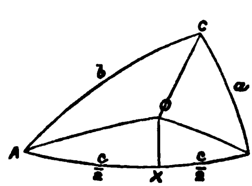
and this meets the directrix $x + 2m = 0$, and makes intercept

$$\frac{-m + 2m\cos^2\alpha}{\sin\alpha\cos\alpha} = 2m\cot 2\alpha;$$

hence $\cot 2\alpha$, $\cot 2\alpha'$, $\cot 2\alpha''$ are in arithmetical progression, and $\tan 2\alpha$, $\tan 2\alpha'$, $\tan 2\alpha''$ in harmonic progression.

7793. (By W. J. MCCLELLAND, B.A.)—Prove that the angles at the centre of the circum-circle of a spherical triangle subtended by the opposite arcs are respectively double of the angles of the chordal triangle.

Solution by B. HANUMANTA RAU, M.A.; *the* PROPOSER; *and others.*



Let ABC be the triangle, and O the centre of the circum-circle, then $\angle OAB = S - C$, and $\cos AOX = \cos\frac{1}{2}c\sin(S - C)$......(1).
We have, if C′ be the angle of the chordal triangle at C,

$$\cos C' = \frac{1 + \cos c - \cos a - \cos b}{4\sin\frac{1}{2}a\sin\frac{1}{2}b}$$

$$= \sin(S - A)\cos\tfrac{1}{2}c \quad ....(2).$$

Equate (1) and (2), hence $\cos AOX = \cos C'$, therefore $\angle AOB = 2C'$; similarly $\angle BOC = 2A'$, and $\angle COA = 2B'$.

7931. (By Professor WOLSTENHOLME, M.A., Sc.D.)—If the sides of a spherical triangle ABC be bisected in a, b, c, and α, β, γ be the arcs bc, ca, ab, and E the spherical excess, prove that

$$\frac{\cos\alpha}{\cos\frac{1}{2}a} = \frac{\cos\beta}{\cos\frac{1}{2}b} = \frac{\cos\gamma}{\cos\frac{1}{2}c} = \cos\tfrac{1}{2}\mathrm{E}.$$

Solution by Professor TANNER, M.A.; J. McDOWELL, M.A.; *and others.*

From the spherical triangle AB′C′ (where B′, C′ are points of bisection),

$$\cos\alpha = \cos\tfrac{1}{2}b\cos\tfrac{1}{2}c + \sin\tfrac{1}{2}b\sin\tfrac{1}{2}c\cos\mathrm{A} = \cos\tfrac{1}{2}\mathrm{E}\cos\tfrac{1}{2}a.$$

[TODHUNTER's *Spherical Trigonometry*, Chap. viii., Ex. 14.]

7960. (By the late Professor CLIFFORD, F.R.S.)—Assuming that

$$\phi(n) = (n+\tfrac{1}{2}f)^2 a + 2(n+\tfrac{1}{2}f)x + ng\pi i, \quad \text{and} \quad \theta^f_g(x) = \Sigma\epsilon^{\phi(n)},$$

the summation extending from $n = -\infty$ to $n = +\infty$, find expressions for $\theta^f_g(x+\frac{1}{2}p\pi i+\frac{1}{2}qa)$ in the two forms $\mathrm{A}\theta(x+\mathrm{B})$ and $\mathrm{C}\theta^r_s(x)$.

Solution by Professor LLOYD TANNER, M.A.

The general terms of $\theta^f_g(x+\frac{1}{2}h\pi i+\frac{1}{2}qa)$, $\mathrm{A}\theta(x+\mathrm{B})$, $\mathrm{C}\theta^r_s(x)$ are

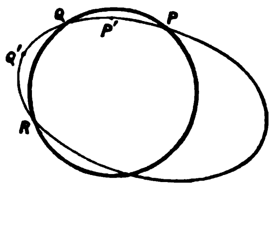
$$e\,|\,(n+\tfrac{1}{2}f)^2a+2(n+\tfrac{1}{2}f)(x+\tfrac{1}{2}p\pi i+\tfrac{1}{2}qa)+ng\pi i\,|,$$

$$e\,|\,\log\mathrm{A}+n^2a+2n(x+\mathrm{B})\,|,$$

and $e\,|\,\log\mathrm{C}+(n+\frac{1}{2}r)^2a+2(n+\frac{1}{2}r)x+ns\pi i\,|$ respectively [$e\,|\,x\,|$ for e^x]. These are equal for all values of n (and, therefore, the θ functions are equal), if $2\mathrm{B} = (f+q)a+(p+g)\pi i$, $4\log\mathrm{A} = f^2a+4fx+2f(p\pi i+qa)$, $r = f+q$, $s = p+g$, $4\log\mathrm{C} = -qa^2-4qx+2fp\pi i$. These seem to be the simplest forms, but there are an infinite number of solutions.

7923. (By Professor CROFTON, F.R.S.)—Show that no circle can meet any given closed convex contour in more than two points, if its radius be greater than the greatest or less than the least radius of curvature of the contour.

Solution by Rev. T. C. SIMMONS, M.A.



Let P, Q, R be three successive points common to the contour and a circle which meets it more than twice.

Then at Q the curves either do, or do not, cross.

If they cross, then, of the two intercepted areas, one such as PQ lies wholly within, and the other QR wholly without, the circle.

Move the circle in its plane so as to

continuously diminish the area PQ, P and Q thereby approaching each other. The area will finally vanish, and P and Q coalesce at some point P′ between P and Q. The circle will now here lie wholly within the contour, showing that at P′ the length of the radius ρ of curvature is $>r$. In a similar way, the circle may be moved so as to make Q and R coalesce at a point Q′ whose radius of curvature will be $<r$.

If the curves do not cross at Q, then either both the areas lie within, or both without, the circle. In the first case, it will be evident, by the same method of proof, that for points to the right and left ρ is $>r$, while at Q itself ρ is $<r$; and *vice versâ* in the second case.

Hence, in order that a circle may meet the contour in three points, its radius must be intermediate between the greatest and least values of ρ.

8001. (By Professor LLOYD TANNER, M.A.)—[Suggested by Mr. WALKER's solution of Quest. 4516, Vol. xli., p. 89.]—In a spherical triangle, prove that, if 3 sides are acute, 2 angles are acute; if 1 side is acute and 1 side is not acute, 1 angle is obtuse and 2 are acute; if 2 sides are obtuse and 1 side is acute, 1 angle is obtuse; if 2 sides are obtuse and the other is not acute, all the angles are obtuse. [The converse group of propositions may be written down by interchanging "angle" with "side," and "acute" with "obtuse," and may be proved from the original group by a purely logical process, or by using polar triangle.]

Solution by EMILY PERRIN.

Let a, b, c, and therefore A, B, C, be in descending order of magnitude; then, by NAPIER's analogy, $\tan\frac{A+B}{2} = \frac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}(a+b)}\cot\frac{c}{2}$; and, as $\cos\frac{1}{2}(a-b)$ and $\cos\frac{1}{2}c$ are essentially positive, $\tan\frac{1}{2}(A+B)$ and $\cos\frac{1}{2}(a+b)$ have the same sign.

Case I.—If a, b, c are all acute, $\cos\frac{1}{2}(a+b)$ and $\cos\frac{1}{2}(a+c)$ are both positive, so therefore $\tan\frac{1}{2}(A+B)$, $\tan\frac{1}{2}(A+C)$ are also positive, $A+B<\pi$ and $A+C<\pi$; therefore the two smaller angles, B, C, are acute.

Case II.—If $a \nless \frac{1}{2}\pi$, and b, c are each $<\frac{1}{2}\pi$, then $\cos a$ is negative, $\cos b$, $\cos c$ are positive; from $\cos a = \cos b\cos c + \sin b\sin c\cos A$, $\cos A$ is negative, therefore A is obtuse. Similarly, from $\cos b = \cos c\cos a + \sin c\sin a\cos B$ and $\cos c = \cos a\cos b + \sin a\sin b\cos C$, $\cos B$, $\cos C$ are positive; therefore B, C are acute.

Case III.—a, b are both $>\frac{1}{2}\pi$ and $c<\frac{1}{2}\pi$; therefore $\cos\frac{1}{2}(a+b)$ is negative, therefore $\tan\frac{1}{2}(A+B)$ is positive, therefore $A+B>\pi$, therefore A is obtuse.

Case IV.—a, b both $>\frac{1}{2}\pi$ and $c \nless \frac{1}{2}\pi$, $\cos\frac{1}{2}(a+b)$ and $\cos\frac{1}{2}(b+c)$ are each negative; therefore $A+B>\pi$ and $B+C>\pi$, as above; therefore A, B at least are obtuse, and $\cos C = \frac{\cos c - \cos a\cos b}{\sin a\sin b}$ = a negative quantity, as $\cos a$, $\cos b$, $\cos c$ are all negative, therefore C is also obtuse.

7954. (By W. J. C. SHARP, M.A.)—In a triangle ABC, if p_1 be the perpendicular from A upon BC, r the radius of the inscribed circle and r_1 that of the escribed circle touching BC; show that (1) $\frac{1}{r} - \frac{1}{r_1} = \frac{2}{p_1}$; (2) the same equation holds if p_1 be the perpendicular from the vertex A of a tetrahedron upon the opposite face, and r the radius of the inscribed sphere, and r_1 that of the sphere touching BCD and the other faces produced. [This may be easily proved without assuming the values of r, &c.]

Solution by W. J. GREENSTREET, B.A.; G. G. STORR, B.A.; *and others.*

$$\frac{1}{r} - \frac{1}{r_1} = \frac{s}{S} - \frac{s-a}{S} = \frac{a}{S} = \frac{a}{\frac{1}{2}ap_1} = \frac{2}{p_1}.$$

6413 & 7151. (By the EDITOR.)—(6413.) A coin of radius r is thrown at random (every possible position being supposed to be equally probable) upon a *rimmed* table whose top is a regular hexagon of in-radius a; show that, if p_n be the probability of the coin's resting on n of the triangles into which the top of the table is divided by its diagonals, then $p_5 = 0$ (always); and (1), when $r < \frac{1}{3}a$, then we shall have

$$p_1 = \frac{(a-3r)^2}{(a-r)^2}, \quad p_2 = \frac{2(2a-5r)\,r}{(a-r)^2}, \quad p_3 = \frac{r^2}{(a-r)^2},$$
$$p_4 = \frac{(1-\frac{1}{6}\pi\sqrt{3})\,r^2}{(a-r)^2}, \quad p_6 = \frac{\frac{1}{6}\pi\sqrt{3}\,.\,r^2}{(a-r)^2};$$

(2) when $r = \frac{1}{3}a$, then $p_1 = 0$, $p_2 = \frac{1}{2}$, $p_3 = \frac{1}{4}$,

$$p_4 = \tfrac{1}{4}(1-\tfrac{1}{6}\pi\sqrt{3}) = \cdot 0233 = \tfrac{1}{43} \text{ nearly},$$
$$p_6 = \tfrac{1}{24}\pi\sqrt{3} = \cdot 2267 = \tfrac{17}{75} \text{ nearly};$$

(3) when $r > \frac{1}{3}a$ and $< \frac{1}{2}a$, then $p_1 = 0$,

$$p_2 = \frac{2(a-2r)^2}{(a-r)^2}, \quad p_3 = \frac{(a-2r)(4r-a)}{(a-r)^2}, \quad \text{and } p_4,\ p_6 \text{ as in (1)};$$

(4) when $r = \frac{1}{2}a$, then $p_1 = p_2 = p_3 = p_5 = 0$, $p_4 = 1-\frac{1}{6}\pi\sqrt{3} = \frac{4}{43}$, $p_6 = \frac{1}{6}\pi\sqrt{3} = \frac{39}{43}$; (5) when $r > \frac{1}{2}a$ and $< 2(2-\sqrt{3})\,a$, *i.e.*, $< \frac{6}{11}a$, that is $> \frac{11}{22}a$ and $< \frac{12}{22}a$, then (putting a_1 for $a-r$), $p_4 = 1-p_6$; p_1, p_2, p_3, p_5 are all zero; and

$$p_6 = \sqrt{3}\left(\frac{r^2}{a_1^2}-1\right)^{\frac{1}{2}} + \sqrt{3}\left(\frac{\pi}{6}-\sec^{-1}\frac{r}{a_1}\right)\frac{r^2}{a_1^2},$$

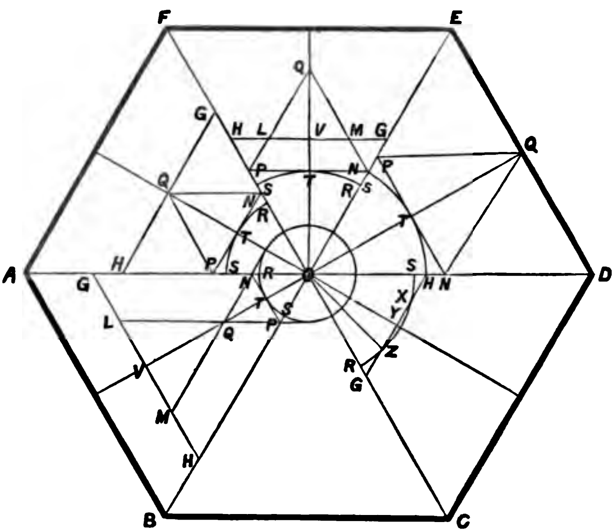
(6) when $r > \frac{6}{11}a$, the coin *must* rest on all six of the triangles;
(7) if the table be *rimless*, the probabilities in (1), and the like in other cases, will be

$$p_1 = \frac{(a-2r)^2}{a^2}, \quad p_2 = \frac{2r}{a^2}(2a-3r)^2, \quad p_3 = \frac{r^2}{a^2},$$
$$p_4 = \frac{r^2}{a^2}\left(1-\frac{\sqrt{3}}{6}\pi\right), \quad p_6 = \frac{\pi r^2\sqrt{3}}{6a^2}.$$

(7151.) A coin is thrown at random upon a plane which is divided into equilateral triangles by three systems of parallel lines; find the respective probabilities of the coin's resting on 0, 1, 2, 3, 4, 5, 6 of the triangles.

Solutions of (6413) *by the* PROPOSER; (7151) *by* D. BIDDLE.

(6413.) Around the centre O of the top ABCDEF of the table, draw a circle RTS of radius r, and draw tangents thereto parallel to the diagonals of the hexagon, and GH parallel to AB at a distance r therefrom.



Then, confining our attention, as involving all possible variations, to the several probabilities while the coin's centre takes every permissible position on the triangle AOB, that is to say, while it moves over the triangle OGH, we see at once that there is no position in which the coin can rest on 5 triangles, and that the respective probabilities of resting on 1, 2, 3, 4, 6 triangles will be the several ratios to the triangle OGH of the following areas : ΔLMQ, Trapezoids GLQN + MHPQ $\equiv$ 2 Trap. GLQN, ΔQNP, Mixtilinear triangles NTR + TPS $\equiv$ 2ΔNTR, Sector ORS.

(1) Now the equilateral triangles in the figure are to each other as the squares of their perpendiculars ; hence we have

$$p_1 = \frac{QV^2}{OV^2} = \frac{(a-3r)^2}{(a-r)^2}, \quad p_2 = \frac{GOH-(LMQ+2ONP)}{GOH}$$

$$= \frac{OV^2-(QV^2+2OT^2)}{OV^2} = \frac{(a-r)^2-[(a-3r)^2+2r^2]}{(a-r)^2} = \frac{2\,(2a-5r)\,r}{(a-r)^2},$$

$$p_3 = \frac{\mathrm{QT}^2}{\mathrm{OV}^2} = \frac{r^2}{(a-r)^2}, \quad p_4 = \frac{(1-\frac{1}{6}\pi\sqrt{3})\,r^2}{(a-r)^2}, \quad p_6 = \frac{\frac{1}{6}\pi\sqrt{3}\,r^2}{(a-r)^2};$$

and the sum of these five fractions is, of course, unity.

(2) When $r = \frac{1}{3}a$ (as shown on the triangle OAF), the point Q coincides with V, the p_1-triangle LMQ vanishes, and then and thereafter the coin cannot rest on *one* triangle, but, with stated probabilities, must rest on 2, 3, 4, or 6 triangles.

(3) When $r > \frac{1}{3}a$, the limit-line GH crosses the p_3-triangle NQP at a distance $(3r-a)$ inside its vertex (as shown on the triangle OFE), the values of p_4, p_6 remain unchanged, their sum being $r^2/(a-r)^2$, and the values of p_2, p_3 will be as hereunder :—

$$p_2 = \frac{2\Delta\mathrm{LHP}}{\Delta\mathrm{OGH}} = \frac{2\mathrm{TV}^2}{\mathrm{OV}^2} = \frac{2\,(a-2r)^2}{(a-r)^2},$$

$$p_3 = \frac{\mathrm{LMNP}}{\Delta\mathrm{OGH}} = \frac{\mathrm{QT}^2-\mathrm{QV}^2}{\mathrm{OV}^2} = \frac{(a-2r)(4r-a)}{(a-r)^2}.$$

(4) When $r = \frac{1}{2}a$, the limit-line GH or NP touches the arc STR (as shown on the triangle OED), the values of p_2, p_3 vanish, and the coin can only rest on either four or six triangles, the probabilities of which are

$$p_4 = 1-\tfrac{1}{6}\pi\sqrt{3} = \cdot 0932 = \tfrac{4}{43} \text{ nearly},$$
$$p_6 = \tfrac{1}{6}\pi\sqrt{3} = \cdot 9068 = \tfrac{39}{43} \text{ nearly}.$$

(5) When $r > \frac{1}{2}a$, the limit-line GH will cut the arc RS in two points (X, Z say, as shown on the triangle OCD) until it has moved up to the chord RS, which will take place when $r+\mathrm{OY} = a$, or $r+\frac{1}{2}r\sqrt{3} = a$, that is to say, when $r = 2\,(2-\sqrt{3})\,a = \cdot 536a = \frac{6}{11}a$ nearly. So long as r is between these limits, we have

$$p_4 = \frac{\text{space RGZ}}{\Delta\mathrm{OGY}}; \quad p_6 = \frac{\Delta\mathrm{OYZ}+\text{sector ORZ}}{\Delta\mathrm{OGY}} \text{ which (putting } a-r \equiv a_1)$$
$$= \sqrt{3}\left(\frac{r^2}{a_1^2}-1\right)^{\frac{1}{2}} + \sqrt{3}\left(\frac{\pi}{6}-\sec^{-1}\frac{r}{a_1}\right)\frac{r^2}{a_1^2}.$$

(6) When G moves up to R, then

$$\sec^{-1}\frac{r}{a_1} = \frac{\pi}{6}, \quad \frac{r}{a_1} = \sec\frac{\pi}{6} = \frac{2}{\sqrt{3}}, \quad \text{and } p_6 = 1.$$

(7) If the coin be thrown upon a *rimless* table whose top is a regular hexagon of in-radius a, one half of the coin may rest over the edge of the table, and the probabilities will be obtained by putting, throughout, in the foregoing results, $(a+r)$ in place of a.

(7151.) Let a = side of triangle, and r = radius of coin. Then, as there will be no possibility of the coin's resting on 0 triangle, unless the plane (or that portion of it which is scored) be limited, we have $p_0 = 0$.

Moreover, it is impossible for any circular disc to rest on *five* triangles, arranged as in the question, without including the apex of a sixth; for all that the disc covers must lie on one side of a line tangential to it, and no such line, however limited, can be drawn between the first and fifth triangles, in the case before us, without crossing a sixth, unless we take a

series of five in a row, when the disc could not possibly extend to the required limits without encompassing other triangles, therefore $p_5 = 0$.

There is a possibility with regard to each of the other sets of triangles named, but only within certain limits. Thus, for one triangle, the disc must not exceed in radius that of the inscribed circle, or one-third of the height of the triangle $= \frac{1}{3}h$; for 2 and 3 triangles, r must not exceed $\frac{1}{3}h$; for 4 triangles, r must not exceed $\frac{2}{3}h$; and for 6 triangles, r must not exceed h. Indeed, to insure a coin resting on no more than on 6 triangles, r must not exceed $\frac{1}{2}h$.

If $r < \frac{1}{3}h$, Fig. 1 will represent the several spaces in which the centre of the coin can lie in order to cover 1, 2, 3, 4, 6 triangles respectively. The size of the coin used on the occasion is apparent from the fact that the spaces marked (6) are each exactly a sixth of its area.

The space marked (1) will correspond with the triangle LMQ in the foregoing, and, as before, we shall obtain

$$p_1 = \frac{(a-2\sqrt{3}r)^2}{a^2}.$$

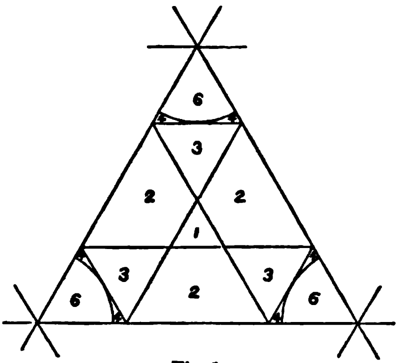


Fig. 1.

Again, the area of each trapezoidal space marked (2) is

$$[(a\sqrt{3}-4r)^2-(a\sqrt{3}-6r)^2]\,/\,4\sqrt{3},$$

therefore $$p_2 = \frac{3\,[(a\sqrt{3}-4r)^2-(a\sqrt{3}-6r)^2]}{3a^2} = \frac{12r\,(a\sqrt{3}-5r)}{3a^2},$$

provided $(a\sqrt{3}-6r)$ have a positive value.

And here we may observe that, in regard to probabilities, no term whose value is below zero is counted; in other words, changes of sign are not allowed. Thus, in the foregoing equation, if $6r > a\sqrt{3}$, then $(a\sqrt{3}-6r)$ is reckoned as 0; and if $4r > a\sqrt{3}$, then the whole becomes *nil;* for the spaces cease to be trapezoidal when the coin is larger than the inscribed circle, and finally vanish when $r > \frac{1}{4}\sqrt{3}\,a$.

The probability as to 3 triangles is the ratio borne to the total area of the given triangle by the sum of the triangular spaces marked (3). When, however, $6r = a\sqrt{3}$, these spaces meet in the centre of the triangle; and when $6r > a\sqrt{3}$, they overlap, and the portions overlapping cease to belong to the three-triangle spaces, and become part of the four-triangle territory (see Fig. 2). Now the area of any one of these smaller triangles is) $\frac{1}{3}r^2\sqrt{3}$, and of the central overlapping portion, when it exists, $(6r-a\sqrt{3})^2/4\sqrt{3}$,

therefore $$p_3 = \frac{3\,[4r^2-(6r-a\sqrt{3})^2]}{4\sqrt{3}} \div \frac{a^2\sqrt{3}}{4} = \frac{3\,[4r^2-(6r-a\sqrt{3})^2]}{3a^2}.$$

When the coin rests on 4 triangles, its centre, except under the circumstances just alluded to, is situated on one of the spaces marked (4). The area of this space is $2r^2(2-\frac{1}{3}\pi\sqrt{3})/4\sqrt{3}$. But account must be taken of the central space which developes when $6r > a\sqrt{3}$. In Fig. 2 it is seen in a partially developed condition; in Fig. 3, when the two-triangle and three-triangle spaces are wholly eliminated. The area of the central

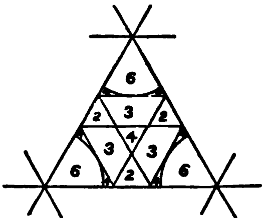


Fig. 2.

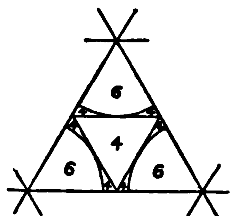


Fig. 3.

space when it forms part of the four-triangle territory, we have seen to be $(6r-a\sqrt{3})^2/4\sqrt{3}$. Therefore

$$p_4 = \frac{6r^2(2-\frac{1}{3}\pi\sqrt{3})+(6r-a\sqrt{3})^2}{4\sqrt{3}} \div \frac{a^2\sqrt{3}}{4} = \frac{6r^2(2-\frac{1}{3}\pi\sqrt{3})+(6r-a\sqrt{3})^2}{3a^2},$$

The six-triangle space, which is found at each corner of the triangle, is exactly one-sixth the area of the coin. The three spaces together, therefore, equal one-half the coin. Therefore

$$p_6 = \frac{r^2\pi}{2} \div \frac{a^2\sqrt{3}}{4} = \frac{2r^2\pi\sqrt{3}}{3a^2}.$$

Consequently, placing the several probabilities side by side, we find $p_0 = 0$,

$$p_1 = \frac{(a\sqrt{3}-6r)^2}{3a^2}, \quad p_2 = \frac{3\left[(a\sqrt{3}-4r)^2-(a\sqrt{3}-6r)^2\right]}{3a^2},$$

$$p_3 = \frac{3\left[4r^2-(6r-a\sqrt{3})^2\right]}{3a^2}, \quad p_4 = \frac{6r^2(2-\frac{1}{3}\pi\sqrt{3})+(6r-a\sqrt{3})^2}{3a^2},$$

$$p_5 = 0, \quad p_6 = \frac{2r^2\pi\sqrt{3}}{3a^2}.$$

And, if we discard all those compound terms which in a given instance have no real value (a rule to be always observed in the estimation of probabilities), then $\Sigma p = 1$, as it ought to be.

It is assumed, in this solution, that the coin may be any size, provided $r < \frac{1}{3}h$.

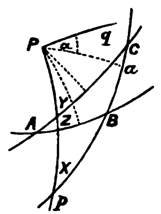
When $4r > a\sqrt{3}$, the four-triangle territory declines in area; but, together with the six-triangle spaces, it makes up the whole given triangle. Therefore, when $4r > a\sqrt{3}$, $p_4 = \frac{3a^2-2r^2\pi\sqrt{3}}{3a^2}$. Consequently the equation to suit all cases should be

$$p_4 = \frac{6r^2(2-\frac{1}{3}\pi\sqrt{3})+(6r-a\sqrt{3})^2-12r(4r-a\sqrt{3})}{3a^2}.$$

7945. (By W. J. McClelland, B.A.)—If through any point P on the surface of a sphere three great circles be drawn cutting the sides of a triangle at angles X, Y, Z; X_1, Y_1, Z_1; X_2, Y_2, Z_2; prove the determinant relation

$$\begin{vmatrix} \cos X, & \cos Y, & \cos Z \\ \cos X_1, & \cos Y_1, & \cos Z_1 \\ \cos X_2, & \cos Y_2, & \cos Z_2 \end{vmatrix} \equiv 0.$$

Solution by Colonel Clarke, C.B., F.R.S.



From P draw perpendiculars, as in the figure, on the sides BC, CA, AB of the triangle; and, denoting by a the angle made by Pa with an arbitrary initial line Pq; θ the angle qPp, determining the great circle through P which cuts the sides at angles X, Y, Z, we have $\cos X = \sin(\theta - a)\cos Pa$, with similar equations for each of the other two great circles which are determined by θ_1 and θ_2.

Put now $\quad \cos a \cos Pa = u, \quad -\sin a \cos Pa = v,$

then the three equations for X, X_1, X_2 are transformed to

$$u \sin\theta + v\cos\theta = \cos X, \quad u\sin\theta_1 + v\cos\theta_1 = \cos X_1,$$
$$u\sin\theta_2 + v\cos\theta_2 = \cos X_2.$$

From these eliminate u, v, and we get the first of the following set of equations, the second and third being provided by the sides C, AB:

$$\cos X\sin(\theta_2-\theta_1) + \cos X_1 \sin(\theta-\theta_2) + \cos X_2 \sin(\theta_1-\theta) = 0,$$
$$\cos Y\sin(\theta_2-\theta_1) + \cos Y_1 \sin(\theta-\theta_2) + \cos Y_2 \sin(\theta_1-\theta) = 0,$$
$$\cos Z\sin(\theta_2-\theta_1) + \cos Z_1 \sin(\theta-\theta_2) + \cos Z_2 \sin(\theta_1-\theta) = 0,$$

and, eliminating from these the three sines, we get the required result.

7863 & 7866. (By Professor Wolstenholme, M.A., Sc.D.)—(7863.) Given a focus and the corresponding directrix of a conic, a circle is drawn touching the axis of the conic at the given focus and intersecting the conic in two points P, Q; prove that, although the straight line PQ depends on two independent parameters (the excentricity of the conic and the radius of the circle), it always touches a certain quartic tricusp, the same curve as is discussed in Quest. 7220 (Vol. 40, p. 114), where it appears in two different characters as an envelope, both distinct from its conditions in this question. If the chord PQ make an angle θ with the axis, the perpendicular upon it from the focus is $c \tan \frac{1}{2}\theta$, where c is the given distance of focus and directrix.

[Professor Wolstenholme thinks this a very peculiar result, but believes that the following fact involves an explanation of it:—Suppose any straight line meets any two of the circles in PQ, P′Q′, the angles POP′, QOQ′ will be equal; and the same if it meet any two of the conics in P, Q; P′, Q′. Certainly, *à priori* it would appear pretty certain that the equation of PQ must involve both the parameters e and b, the excentricity of the conic and the radius of the circle, and might, therefore, be made to coincide with *any* straight line. Such argument is generally valid, and

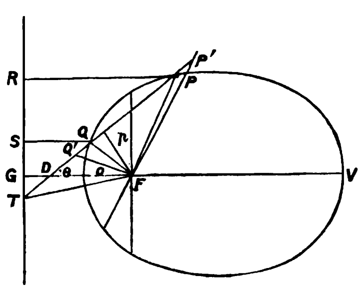
it is interesting to discover the reason of any exception. The curve of this question is completely defined and its equation found in the answer to Quest. 7220, but it may also be generated by taking the inverse of a rectangular hyperbola with respect to a vertex; then the first negative polar of this inverse with respect to *its* vertex is the quartic tricusp in question. It may be generated in an infinite number of ways as an envelope, and perhaps may be taken as Protean a locus.

(7866.) A parabola has a given focus S, and a given direction of axis; a circle has its centre at a fixed point O on the latus rectum of the parabola; prove that the points of intersection of their common tangents lie on a fixed nodal circular cubic having its node at O, its vertex at S, and its asymptote parallel to the axis of the parabolas, and at a distance 2SO. Explain how there comes to be a definite locus when we have *two* variable parameters (the radius of the circle and the latus rectum of the parabola).

[The equation of the locus in 7866 is (1), referred to polar coordinates with S for pole, $r = c\tan\frac{1}{2}\theta$ or $r = c\cot\frac{1}{2}\theta$, which two equations represent the same curve; (2) referred to rectangular coordinates with O for origin, and OS for axis of x, $y^2 = x^2\dfrac{a-x}{a+x}$, where OS $= a$. This well-known circular cubic is the inverse of a rectangular-hyperbola with respect to a vertex, and the pedal of a parabola with respect to the foot of the directrix.

Generalized by Projection, the theorem is as follows:—A conic U is inscribed in a given triangle ABC so as to touch BC in a fixed point a, and a' is the point on BC harmonically conjugate to a. On Aa' is taken a fixed point O and a second conic V described touching OB, OC at B and C; prove that the points of intersection of common tangents to any two such conics lie on a fixed cubic having a node at O, touching Aa at A, passing through B, C, a, and whose tangent at a meets AO in a point which divides Oa' harmonically to a. Also, explain how such points can have a definite locus when we have *two* variable parameters (one for each conic) to deal with. Of course, the whole locus might be obtained from any one conic U by varying V alone; or from any one conic V by varying U alone. By reciprocating this, we get an envelope remarkable in the same way, as depending on *two* variable parameters.]

Solution by Arthur Hill Curtis, LL.D., D.Sc.



F being the focus of the conic in Question 7863, let PR, QS, and FG be the perpendiculars from P, Q, and F on the directrix RG, let PQ intersect the directrix in T, then $\dfrac{PT}{QT} = \dfrac{PR}{QS} = \dfrac{FP}{FQ}$, therefore FT is the bisector of the external vertical angle of $\triangle$PFQ, therefore

$\angle PTF = \frac{1}{2}(\angle FQP' - \angle FPQ)$,

or, as FG is tangent to the circle circumscribing $\triangle$PFQ,

$$= \tfrac{1}{2}(\angle FQP - \angle QFG) = \tfrac{1}{2}\angle QDF,$$

and therefore p, the perpendicular from F on PQ,

$$= \text{TF} \sin \tfrac{1}{2}\theta = \text{TF} \cos \tfrac{1}{2}\theta \tan \tfrac{1}{2}\theta, = c \tan \tfrac{1}{2}\theta.$$

The equation of PQ therefore is $x \sin \theta + y \cos \theta - c \tan \frac{1}{2}\theta = 0$...........(1), or $2x \tan \frac{1}{2}\theta + y\,(1 - \tan^2 \frac{1}{2}\theta) - c \tan \frac{1}{2}\theta\,(1 + \tan^2 \frac{1}{2}\theta) = 0$; or, putting $\tan \frac{1}{2}\theta = \mu$, $y + \mu\,(2x - c) - \mu^2 y - \mu^3 c = 0$; or, changing origin, by substituting $2x$ for $2x - c$, $y + 2\mu x - \mu^2 y - \mu^3 c = 0$. Now the envelope of $L + \mu M + \mu^2 N + \mu^3 R = 0$, is $4\,(N^2 - 3RM)(M^2 - 3LN) - (MN - 9LR)^2 = 0$, and therefore the envelope required is $4\,(y^2 + bcx)(4x^2 + 3y^2) - y^2\,(9c - 2x)^2 = 0$. The curve referred to can be identified with the above by putting $a = -\frac{1}{4}c$.

Equation (1) shows that the line PQ depends on only *one* parameter θ, and does not vary with the excentricity of the ellipse, and that the envelope will be the same whatever conic of the system is selected as defining it. This also appears thus—Let any other conic of the system cut the line PQ in P′, Q′, then, as FT is the common bisector of external vertical angle in Δ's PFQ, P′FQ′, $\angle$ PFP′ = $\angle$ QFQ′, and therefore (as circle circumscribing ΔPFQ touches FG, and consequently $\angle$ QFG = $\angle$ QPF), $\angle$ Q′FG = $\angle$ Q′P′F, and circle circumscribing ΔP′FQ′ also touches FG; or thus—it is plain that, if P be assumed anywhere, the conic of the system passing through P is determinable, and the corresponding point Q on it is then found by the condition $\angle$ PQF = $\angle$ PFV; so that the line PQ cannot be *any* straight line—involving *two* parameters.

Again, as $p = c \tan \frac{1}{2}\theta$, the envelope considered is the negative pedal of the locus $r = c \tan \frac{1}{2}\theta$, which gives

$$x = r \sin \theta = 2c \sin^2 \frac{\theta}{2} = c\,(1 - \cos \theta) = c\left(1 - \frac{y}{r}\right),$$

or $$c - x = \frac{cy}{r}, \text{ or } (c - x)^2 = \frac{c^2 y^2}{x^2 + y^2}, \text{ or } (c - x)^2 x = (2c - x)\,y^2,$$

or, putting $c - x = X$, thereby changing origin from F to O′, through distance c along axis of a,

$$X^2 (c - X) = y^2\,(c + X), \quad \text{or} \quad y^2 = \frac{X^2\,(c - X)}{(c + X)},$$

which may also be written $c\,(X^2 - y^2) - X\,(X^2 + y^2) = 0$, a circular cubic, whose vertex is at origin and whose equation in polar coordinates is $(\cos^2 \phi - \sin^2 \phi) - \dfrac{r}{c} \cos \phi = 0$, which can easily be identified with the pedal of a parabola (parameter $= 4c$) with regard to foot of directrix, and the inverse of which is $\cos^2 \phi - \sin^2 \phi - \dfrac{k^2}{cr} \cos \phi = 0$,

or $$X^2 - y^2 - \frac{k^2}{c} X = 0, \quad \text{or} \quad \left(X - \frac{k^2}{2c}\right)^2 - y^2 = \left(\frac{k^2}{2c}\right)^2,$$

an equilateral hyperbola, whose vertices are at O′ and at the point along axis of X distant from O′ by $\dfrac{k^2}{c}$, that is, the inverse point to F.

As p is defined by $p = c \tan \frac{1}{2}\theta$, the inverse of the locus of the extremity of p, which is also the reciprocal polar of the envelope of the line PQ, will have for equation $r \equiv \dfrac{k^2}{p} = \dfrac{k^2}{c} \cot \dfrac{\theta}{2} = a \cot \dfrac{\theta}{2}$, which may also be written $r = a \tan \frac{1}{2}\phi$, where ϕ is the supplement of θ,—the circular cubic already considered c, and θ being replaced by a and ϕ, while, by recipro-

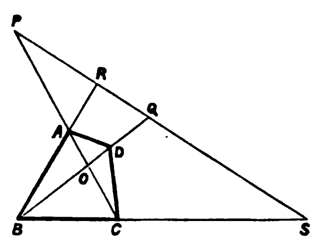
cating the theorem, this curve will be the locus of the intersection of the common tangent to a circle whose centre is on FG at the point O where $FO = \frac{k^2}{c}$, and into which the ellipse reciprocates, and a parabola whose focus is at F and whose axis is perpendicular to FO, and into which the circle touching FO at F reciprocates. Now, when $r = \infty$ in the curve $r = a \tan \frac{1}{2}\phi$, $\phi = \pi$, and, if for $\phi = \pi$ we obtain the corresponding value of P, the perpendicular from origin on tangent, by the formula

$$P = \frac{r^2 d\theta}{[dr^2 + r^2 d\theta^2]^{\frac{1}{2}}} = \frac{1}{\left[\left\{\frac{d}{d\theta}\left(\frac{1}{r}\right)\right\}^2 + \frac{1}{r^2}\right]^{\frac{1}{2}}}, \text{ we obtain } P = 2a \equiv 2FO.$$

As the line PQ, in 7863, really depends only on the parameter c, so the point into which it reciprocates, and whose locus is sought in 7866, only depends on $\frac{k^2}{c}$, or a. The letter F in the figure corresponds to S in Question 7866.

7934. (By W. S. McCay, M.A.)—Prove that the locus of a point at which a given system of four points can be placed in perspective with another fixed system of four points is a conic (in a plane).

Solution by Rev. T. C. Simmons, M.A.; J. O'Regan; *and others.*



Let the fixed system be ABCD, and let AC, BD meet in O. Then, if A′, B′, C′, D′ be corresponding points, in the other system, take P in CA produced and Q in BD produced, so that

$$[COAP] = C'O' : O'A'$$

and $$[BODQ] = B'O' : O'D',$$

and let BA, BC meet the line PQ in R and S. In any plane through PS describe on PQ, RS circular segments containing angles respectively equal to A′O′D′ and A′B′C′; the intersection V of these segments will be one position of the required point.

Also, drawing VM perpendicular on PS, it is evident that, for all positions of the plane, M is fixed and VM constant.

Therefore the required locus is a *circle*.

[The Proposer remarks that the above is a solution to the question "To find the locus of the centre of *projectivity* of two homographic systems, one being in a fixed plane," whereas the proposed question was intended to mean, "Given two sets of four points in a common plane, one set being fixed in position, to find the locus of a point at which the second set could be put in *simple perspective.*" There is, of course, a unique point in the plane at which they are projectives (see Salmon's *Higher Curves*, § 330, and Townsend's *Modern Geometry*, Vol. II., p. 336.]

7939. (By H. Ll. Smith, M.A.)—A district containing $2n$ Liberal and n Conservative voters is divided into three equal wards, each returning one member. Show that, if n be odd, the chance of one Conservative being returned is $3(n+3)/4(n+2)$.

Solution by the Proposer.

Consider one of the wards. It may contain 0, 1, 2, ... n Conservatives. And number of ways in which it may contain r Conservatives = number of ways in which the remaining $(n-r)$ may be divided between the other two wards $= (n-r+1)$; hence the number of ways in which the Conservatives can be distributed in the first ward $=\Sigma_0^n (n-r+1) = \frac{1}{2}(n+1)(n+2)$; and the number of ways in which they may be in a majority is

$$\Sigma_{\frac{1}{2}(n+1)}^n (n-r+1) = \tfrac{1}{8}(n+1)(n+3);$$

hence the chance of a Conservative being returned in the given ward is $\dfrac{n+3}{4(n+2)}$; therefore the chance of a Conservative being returned in one of the three wards is $\dfrac{3(n+3)}{4(n+2)}$.

7957. (By Rev. T. C. Simmons, M.A.)—Show that, from the equations $x^2-yz=a^2$, $y^2-zx=b^2$, $z^2-xy=c^2$, the values of x, y, z are

$$x=\frac{a^4-b^2c^2}{(a^6+b^6+c^6-3a^2b^2c^2)^{\frac{1}{2}}},$$

$$y=\frac{b^4-a^2c^2}{(a^6+b^6+c^6-3a^2b^2c^2)^{\frac{1}{2}}},\quad z=\frac{c^4-a^2b^2}{(a^6+b^6+c^6-3a^2b^2c^2)^{\frac{1}{2}}}.$$

Solution by (1) G. G. Storr, B.A., *and others*; (2) *the* Proposer.

1. Multiplying &c., we obtain $a^2y+b^2z+c^2x=0,\quad a^2z+b^2x+c^2y=0,$

therefore
$$\frac{x}{a^4-b^2c^2}=\frac{y}{b^4-c^2a^2}=\frac{z}{c^4-a^2b^2}\equiv\frac{1}{\lambda}\ (\text{say});$$

and, substituting in any equation, we find $\lambda^2=a^6+b^6+c^6-3a^2b^2c^2$, whence x, y, z are known.

2. Otherwise: we have $a^2x+b^2y+c^2z=(a^4-b^2c^2)x^{-1}$,

$$b^2x+c^2y+a^2z=0,\quad c^2x+a^2y+b^2z=0;$$

therefore
$$\begin{vmatrix} a^2-(a^4-b^2c^2)x^{-2}, & b^2, & c^2 \\ b^2, & c^2, & a^2 \\ c^2, & a^2, & b^2 \end{vmatrix}=0,$$

giving
$$(a^4-b^2c^2)^2x^{-2}=a^6+b^6+c^6-3a^2b^2c^2,$$

and similarly for y and z. (See Vol. xlii., p. 43.)

[At the end of the Appendix to Vol. xlii., there is another solution of this question, which, though every equation is correct, arrives at

the false conclusion that x, y, z are *variable*. The error, however, nowise invalidates the *method* which the solution was intended to illustrate. For, substituting in $a^2y+b^2z+c^2x=0$ the values $y=x-(a^2-b^2)/S$, $z=x-(a^2-c^2)/S$, we at once obtain

$$\frac{x}{a^4-b^2c^2}=\frac{1}{S(a^2+b^2+c^2)}=\frac{y}{b^4-c^2a^2},=\frac{z}{c^4-a^2b^2}\text{ by symmetry;}$$

after which the solution may be completed as in the first method given above.]

7948 & 7951. (By ÂSÛTOSH MUKHOPÂDHYÂY.)—(7948.) Tangents are drawn to any central conic, so that the squares of the intercepts on the minor axis are in arithmetical progression; show that the squares of the sines of the angles which the tangents make with the minor axis are in harmonic progression.

(7951.) Tangents are drawn to a parabola, so that the intercepts they make on the latus rectum produced are in arithmetical progression: prove that the sines of double the angles of inclination of the tangents to the axis are in harmonic progression.

Solution by Rev. J. L. KITCHIN, M.A.; R. KNOWLES, B.A; *and others.*

(7948.) Take the ellipse, the tangent is $\frac{xx'}{a^2}\pm\frac{yy'}{b^2}=1$..............(1);

then $\tan\theta=\mp\frac{b^2x'}{a^2y'}$, therefore $1+\frac{a^2}{b^2}\tan^2\theta=\frac{b^2}{y_1^2}$,

$b^2+a^2\tan^2\theta=\frac{b^4}{y_1^2}$ = squared intercept of (1) on minor axis, therefore squared intercepts are $b^2+a^2\tan^2\theta_1$, $b^2+a^2\tan^2\theta_2$, $b^2+a^2\tan^2\theta_3$, &c., therefore $\tan^2\theta_1+\tan^2\theta_3=2\tan^2\theta_2$ therefore $\sec^2\theta_1+\sec^2\theta_3=2\sec^2\theta_2$, whence $\cos^2\theta_1$, $\cos^2\theta_2$, $\cos^2\theta_3$ are in H. P., &c., or, if $\phi_1=\frac{1}{2}\pi-\theta_1$, $\sin^2\phi_1$, $\sin^2\phi_2$, $\sin^2\phi_3$ are in H. P.

(7951.) This follows in precisely the same way.

SUR LES CERCLES DE TUCKER. *By* Professor NEUBERG.

Soient ABC un triangle quelconque, AD, BE, CF les hauteurs qui se coupent en P, O le centre du cercle ABC, $A_1B_1C_1$ le triangle que forment les tangentes menées en A, B, C au cercle O. On sait que les droites AA_1, BB_1, CC_1 se coupent au point de Lemoine K du triangle ABC.

1. *Si les côtés du triangle* A′B′C′ *sont parallèles à ceux de* ABC *et que les sommets sont sur les symédianes* AK, BK, CK, *les six points* X, Y′, X′, Z, Y, Z′, *où se coupent les côtés des deux triangles* ABC, A′B′C′, *appartiennent à un cercle T dont le centre est sur la droite* KO, *au milieu de la distance des centres des cercles* ABC, A′B′C′. (BC coupe A′B′ en X et A′C′ en X′; CA coupe B′C′ en Y et B′A′ en Y′; AB coupe C′A′ en Z et C′B′ en Z′.)

Cette proposition a été signalée par M. Lemoine au *Congrès de Lyon* (1873) dans une communication intitulée "*Sur quelques propriétés d'un point remarquable d'un triangle*" (voir spécialement le No. XII.). Nous l'avons également fait connaître dans *Mathesis*, t. I. (1881), pp. 15, 59, 187. M. Tucker l'a trouvée de son côté, sans connaître les travaux antérieurs ; voir "*A Group of Circles*," *Quarterly Journal* (Vol. XX., No. 77).

Nous proposons d'appeler les cercles T "*Cercles de Tucker.*" Ils sont susceptibles de deux autres définitions.

2. Les droites ZY′, X′Y, XZ′ sont parallèles aux côtés des triangles DEF, $A_1B_1C_1$, et forment un triangle $\alpha\beta\gamma$ dont les sommets sont sur les symédianes de ABC. Donc:

Si les côtés d'un triangle $\alpha\beta\gamma$ sont parallèles à ceux du triangle orthocentrique DEF *de* ABC *et que les sommets sont sur les symédianes de* ABC, *les côtés non homologues des triangles* ABC, $\alpha\beta\gamma$ *se coupent en six points d'un cercle de Tucker.*

Cette manière de considérer les cercles de Tucker, indiquée dans notre note "*Sur le centre des médianes antiparallèles*" (*Mathesis*, 1881, p. 188), a été étudiée par M. Lemoine (*Mathesis*, 1884, p. 201).

3. Les six points X, X′, Y, Y′, Z, Z′ d'un cercle de Tucker sont les sommets de deux triangles XYZ, X′Y′Z′, égaux entre eux et semblables à ABC. Par conséquent :

Si à un triangle ABC, *on inscrit deux triangles* ZXY, Y′Z′X′, *semblables à* ABC *et tels que les côtés font avec leurs homologues de* ABC *le même angle* ϕ, *les sommets des triangles* XYZ, X′Y′Z′ *appartiennent à un cercle de Tucker.*

Les cercles AZY, BZX, CXY se coupent en un même point ω tel que les angles $Z\omega X = A\omega B = \pi - B$, $X\omega Y = B\omega C = \pi - C$; donc ω *est le premier point de Brocard de* ABC *et* ZXY. De même, *les triangles* ABC, Y′Z′X′ *ont même deuxième point de Brocard* ω'.

Ces propriétés, indiquées, en partie, par M. Tarry et nous (*Mathesis*, 1881, p. 187, §12 ; 1882, p. 73), ont été étudiées à un point de vue général par M. Taylor. [Voir aussi "*The Triplicate-Ratio Circle,*" *by R. Tucker*, dans l'*Appendix to the Proceedings of the London Mathematical Society*, Vol. XIV., No. 214, p. 319, (21).] MM. Taylor et Lemoine ont trouvé la propriété remarquable que *l'enveloppe des cercles de Tucker est la conique qui touche les côtés de* ABC *aux pieds des symédianes et qui a pour foyers les points de Brocard.* ("*The Relations of the Intersections of a Circle with a Triangle*" *by Mr. Taylor*, dans les *Proceedings of the London Mathematical Society*, Vol. XV.)

Appelons *faisceaux de Brocard* les deux triples de droites (ωA, ωB, ωC), (ω'A, ω'B, ω'C). Le troisième mode de génération des cercles de Tucker peut alors être énoncé ainsi :

Si l'on fait tourner les deux faisceaux de Brocard autour de leurs centres d'un même angle ϕ *et en sens contraires, les rayons rencontrent les côtés correspondants de* ABC *en six points d'un cercle de Tucker.*

Nous n'insisterons pas sur les conséquences importantes qui peuvent se tirer de la considération de tels faisceaux de trois rayons égaux, que l'on fait tourner autour de leurs centres.

4. A ces définitions des cercles de Tucker correspondent trois cas particuliers remarquables.

Dans le §1, si les droites XY′, YZ′, ZX′ passent par K, le cercle T prend le nom de *cercle de Lemoine*, du nom du géomètre qui l'a étudié pour la première fois. (*Congrès de Lyon*, 1873 ; *Congrès de Lille*, 1874 ; *Nouvelles Annales*, 1873, p. 264 ; *Mathesis*, 1881, p. 189.) M. Tucker a également

trouvé les propriétés principales de ce cercle qu'il a appelé *Triplicate-Ratio Circle*.

Le centre du cercle de Lemoine est au milieu de KO; les droites $\beta\gamma$, $\gamma\alpha$, $\alpha\beta$ passent par les milieux de AK, BK, CK; l'angle ϕ ou YXC est égal à l'angle de Brocard.

5. Si les droites $\beta\gamma$, $\gamma\alpha$, $\alpha\beta$ passent par K, le triangle A′B′C′ est symétrique de ABC par rapport à K. Le centre du cercle XYZ est en K, et l'angle $\phi = \frac{1}{2}\pi$. Ce cercle a été également signalé par M. Lemoine (*loc. cit.*, et *Nouvelle Correspondance*, t. iii., p. 188, Brocard). M. Casey (*A Sequel to Euclid*) attribue la découverte de ce cercle à M. McCay; il propose la dénomination de "*Cosine Circle*," parce que les droites XX′, YY′, ZZ′ sont proportionnelles à cos A, cos B, cos C.

6. Les symédianes AK, BK, CK passent par les milieux α, β, γ des côtés du triangle orthocentrique DEF. Les côtés du triangle $\alpha\beta\gamma$ étant parallèles à ceux de DEF, ils rencontrent (§2) les côtés de ABC en six points F′, E′, D″, E″, E‴, D‴ d'un cercle, que les géomètres anglais appellent *cercle de Taylor*. ($\beta\gamma$ coupe AB en D″ et AC en D‴; $\gamma\alpha$ coupe BC en E‴ et BA en E′; $\alpha\beta$ coupe CA en F′ et CB en F″.)

Les côtés de ABC étant les bissectrices extérieures des angles de DEF, il est facile de voir que $\alpha F' = \alpha E = \alpha F = \alpha E'$; donc le cercle qui a pour diamètre EF passe par E′ et F′, et EE′, FF′ sont perpendiculaires à AB, AC. Par conséquent, *les pieds des hauteurs des triangles* AEF, BDF, CDE *donnent six points d'une circonférence*. Ce dernier théorème, démontré dans les "Théorèmes et Problèmes, par Catalan" a été proposé aux lecteurs du *Journal de Vuibert;* nous ignorons à qui il est dû. [Comparer aussi *Nouvelle Correspondance Mathématique*, t. VI. (1880), p. 183.]

Le centre I *du cercle de Taylor coïncide avec le centre du cercle inscrit à* $\alpha\beta\gamma$, point qui est le centre de gravité du périmètre du triangle DEF. Cette propriété a même lieu dans le cas général du §2.

Soient P′, P″, P‴, J les orthocentres des triangles AEF, BDF, CDE, DEF. *Le point* I *est au milieu des quatre droites* DP′, EP″, FP‴, OJ, ainsi que nous l'avons fait remarquer dans *Mathesis*, t. I., pp. 14 et 190. En effet, des parallélogrammes PFP′E, PFP″D, PDP‴E, on conclut facilement que les triangles DEF, P′P″P‴ ont leurs côtés égaux et parallèles, et admettent un centre de symétrie I′. Les droites AP, BP″, CP‴, perpendiculaires aux côtés des triangles DEF, P′P″P‴, se coupent en un point qui est à la fois l'orthocentre de P′P″P‴ et le centre du cercle circonscrit à ABC; I′ est donc aussi au milieu de la distance OJ des orthocentres J, O, des triangles DEF, P′P″P‴. Enfin, I′ coïncide avec I; car les lignes αI′, βI′, étant parallèles à FP″, FP′, sont les bissectrices des angles α, β du triangle $\alpha\beta\gamma$. (Comparez *Ed. Times*, Question 7900.)

7. La figure précédente peut être envisagée à deux autres points de vue. Si l'on considère $\alpha\beta\gamma$ comme étant le triangle *primitif*, on a le théorème suivant que nous avons proposé aux lecteurs de *Mathesis* (1881, p. 14): *On prolonge les côtés des angles* α, β, γ *d'un triangle* $\alpha\beta\gamma$ *des quantités*

$$\alpha F' = \alpha E' = \beta\gamma, \quad \beta D'' = \beta F'' = \alpha\gamma, \quad \gamma E''' = \gamma D''' = \alpha\beta;$$

démontrer que les points F′, E′, D″, F″, E‴, D‴ *sont sur une même circonférence concentrique avec le cercle inscrit à* $\alpha\beta\gamma$.

8. Regardons maintenant DEF comme étant le triangle primitif. Les points A, B, C seront les centres des cercles exinscrits à DEF. Soient A′ le pied de la hauteur AP′A′, et m, n les points de rencontre DF, DE avec

E′F′. De ce que l'angle A′F′E = AF′E′ = nF′E, et F′EA′ = DEC = F′En, on conclut facilement que An est perpendiculaire à DE ; par analogie, Am est perpendiculaire à DF. Donc mn est la corde des contacts de l'angle FDE avec le cercle exinscrit A.

De là le théorème suivant :

Les polaires des sommets d'un triangle DEF *par rapport aux cercles exinscrits opposés rencontrent, respectivement, les bissectrices extérieures des angles de* DEF *en six points d'une même circonférence. Ces polaires forment un triangle* $\alpha'\beta'\gamma'$ *dont le centre du cercle circonscrit coïncide avec l'orthocentre* J *de* DEF.

La dernière propriété résulte de ce que DD‴, DD″ sont perpendiculaires à α'D‴, α'D″, et que, par suite, α'D est perpendiculaire à D″D‴ et à FE.

Si l'on observe que la ligne D″D‴ est équidistante des points α' et A, et des points D et A′, on trouve Dα' = AA′. Donc *les distances* Dα', Eβ', Fγ' *sont égales aux rayons des cercles exinscrits* à DFE.

On a $$J\alpha' = JD + D\alpha' = P'O + AA' = AO + P'A'.$$

Mais P′A′ est égal à la distance de P à FE, PFP′E étant un parallélogramme ; par conséquent, *le rayon du cercle circonscrit à* $\alpha'\beta'\gamma'$ *est égal à la somme du diamètre du cercle circonscrit à* DEF, *et du rayon du cercle inscrit à* DEF.

Les propositions du no. 8 ont fait l'objet du concours d'agrégation des Lycées Français en 1873. On en trouve une démonstration trigonométrique par M. Gambey, dans les *Nouvelles Annales*, 1874, p. 43 ; une démonstration géométrique par nous dans la *Nouvelle Correspondance*, t. i., p. 44, et une démonstration analytique par M. Greiner dans les *Archives de Grunert-Hoppe*, t. lxi., p. 225.

M. Taylor a donné des expressions remarquables du rayon du cercle D″E‴F′ et de l'angle ϕ = D″F′A :

$$ID'' = R\,[\sin^2 A \sin^2 B \sin^2 C + \cos^2 A \cos^2 B \cos^2 C]^{\frac{1}{2}},$$

$$\operatorname{tg}\phi = -\operatorname{tg} A \operatorname{tg} B \operatorname{tg} C = -(\operatorname{tg} A + \operatorname{tg} B + \operatorname{tg} C).$$

La 1re peut se déduire de ce que D′D‴ est égal au demi-périmètre de DEF, et que la distance de I à D″D‴ est égal à la moitié du rayon du cercle inscrit à DEF.

9. Dans une Note insérée dans l'*Appendix to Proceedings of the London Mathematical Society*, Vol. xv., M. Tucker indique le théorème suivant :

Si D_1, E_1, F_1 *sont les milieux des côtés d'un triangle* ABC, *les droites de Simson de ces points relatives au triangle orthocentrique* DEF *passent par le centre* I *du cercle de Taylor ; les droites de Simson de* D, E, F *par rapport au triangle* $D_1E_1F_1$ *passent par le même point* I.

Cette proposition avait déjà été trouvée par M. Edm. Van Aubel (*Mathesis*, 1881, p. 207), et elle a été démontrée par M. Liénard (*Mathesis*, 1885). Elle peut être établie par des considérations géométriques très simples. Soient D_2, E_2, F_2 les milieux de AP, BP, CP. La droite D_1D_2 est perpendiculaire au milieu a de EF : BC étant la bissectrice extérieure de l'angle FDE, la droite de Simson de D_1 par rapport au triangle DEF passe par a et est perpendiculaire à BC ; donc elle coïncide avec aI, etc. [See Miss Scott's solution of Quest. 7938.]

La droite de Simson du point D par rapport au triangle $D_1E_1F_1$ passe par le milieu de la hauteur AD (projection de D sur E_1F_1) et par le milieu de DO (O étant l'orthocentre de $D_1E_1F_1$) ; par suite, elle est parallèle à AO et passe par le milieu I de DP′, etc.

M. VAN AUBEL a aussi considéré les droites de Simson des points D_2, E_2, F_2 par rapport au triangle DEF. *Ces lignes passent par α, β, γ et sont parallèles à* BC, CA, AB; *elles forment un triangle dont* I *est l'orthocentre, et dont le centre de similitude par rapport à* ABC *est le centre de gravité de* ABC.

4139. (By Professor SYLVESTER, F.R.S.)—Given

$$x+yu = a(z+tu),\quad xu+y = b(zu+t),\quad x+yv = c(z+tv),$$
$$xv+y = d(zv+t),\quad x+y = e(z+t);$$

determine the relation between a, b, c, d, e; and hence prove that the condition of a quintic $(\alpha, \beta, \gamma, \delta, \epsilon, \theta)(x, y)^5$, being linearly transformable into a recurrent equation, is expressible by a homogeneous symmetric function of the 18th order in the coefficients $\alpha, \beta, \gamma, \delta, \epsilon, \theta$.

Solution by W. J. C. SHARP, M.A.

The given equations are the conditions that a, b, c, d, e should be linearly transformable into $u, \frac{1}{u}, v, \frac{1}{v}$, 1 respectively, and, by eliminating u and v, they lead to

$$y^2-x^2-(ty-zx)(a+b)+ab(t^2-z^2) = 0,$$
$$y^2-x^2-(ty-zx)(c+d)+cd(t^2-z^2) = 0,$$
$$y^2-x^2-(ty-zx)\,2e+e^2(t^2-z^2) = 0,$$

$$\therefore\quad \begin{vmatrix} 1, & a+b, & ab \\ 1, & c+d, & cd \\ 1, & 2e, & e^2 \end{vmatrix} = 0, \text{ or } (b-e)(d-e)(c-a)+(a-e)(c-e)(d-b) = 0.$$

While e is still transformed into 1, two similar conditions are necessary to meet the cases where (1) a and c, and b and d transform into reciprocals, and where (2) a and d, and b and c do so. So that the condition that the quintic whose roots are a, b, c, d, e should transform into a recurring quintic (e transforming into 1) is of the 9th order in the roots, and the full condition is the 45th order in the roots, and of the $3\times 2+12 = 18$th in each root, and therefore in the coefficients. As it is of an odd order in the differences of the roots, it changes sign with the modulus of transformation, and, as it is a symmetrical function of the differences of the roots, it is a skew invariant, viz. I. (See SALMON's *Higher Algebra*, p. 189.)

It is interesting to pursue Professor SYLVESTER's hint as to how to determine the conditions which must be fulfilled, in order that an equation may be transformable into a recurring equation.

If $x+yu = a(z+tu)$, $xu+y = b(zu+t)$, $x+yv = c(z+tv)$, $xv+y = d(zv+t)$, $x+yw = e(z+tw)$, $xw+y = f(zw+t)$, these are the conditions that a, b, c, d, e, and f should be linearly transformable into $u, \frac{1}{u}, v, \frac{1}{v}, w$, and $\frac{1}{w}$, by the same transformation, and eliminating u, v, and w,

$$y^2-x^2-(ty-zx)(a+b)+ab(t^2-z^2) = 0,$$
$$y^2-x^2-(ty-zx)(c+d)+cd(t^2-z^2) = 0$$
$$y^2-x^2-(ty-zx)(e+f)+ef(t^2-z^2) = 0,$$

$$\therefore \quad \begin{vmatrix} 1, & a+b, & ab \\ 1, & c+d, & cd \\ 1, & e+f, & ef \end{vmatrix} = 0\,; \text{ or } (a-e)(b-c)(d-f)+(a-f)(b-d)(c-e) = 0,$$

and, a and b still transforming into reciprocals, there will be two similar conditions, three in all, while a and b transform into reciprocals, as many for a and c and so on, in all five sets of three, so that the symmetrical condition that the sextic equation whose roots are a, b, c, d, e, and f should transform into a recurring equation, is of the 45th order in the roots and 15th order in each root, and therefore in the coefficients, and, being of an odd order in differences of three roots, it is a skew invariant. It is, in fact, Prof. Cayley's invariant E, the vanishing of which is the condition that the roots should form a system in involution, as is indeed necessary from the above equations, which involve the condition that one of the systems of equations of the type $x^2+(a+b)\,x+ab = 0$, $x^2+(c+d)\,x+cd = 0$, and $x^2+(e+f)\,x+ef = 0$ should hold good.

In all cases the conditions that a binary equation should be linearly transformable into a recurring equation, will be obtained by forming the symmetrical conditions for the vanishing of

$$\begin{vmatrix} 1, & 1, & 1 \\ a+b, & c+d, & 2l \\ ab, & cd, & l^2 \end{vmatrix} \text{ or of } \begin{vmatrix} 1, & 1, & 1 \\ a+b, & c+d, & l+m \\ ab, & cd, & lm \end{vmatrix},$$

according as the equation is of odd or even degree, and in general the conditions are $n-2$ in number for an equation of the $(2n-1)^{\text{th}}$ or $2n^{\text{th}}$ order.

For the septic, if a and b, c and d, e and f transform into reciprocals,

and g into 1, $\qquad (a-c)(b-e)(d-f)+(a-f)(b-d)(c-e) = 0,$

and $\qquad (b-g)(d-g)(c-a)+(a-g)(c-g)(d-b) = 0,$

the symmetrical forms of which are an invariant of the 315^{th} order in the roots and 90^{th} in the coefficients, and one of the 945^{th} order in the roots and of the 270^{th} in the coefficients; both, being of odd order in the differences, are skew invariants. The vanishing of these is not, however, a sufficient, though a necessary condition, for the possibility of the transformation, as it does not follow that the vanishing factors will correspond to the same transformation.

Note on Question 7695; *by* C. L. Dodgson, M.A.

The solution given to this question on p. 75 of Vol. 42, is one of the most curious instances I have met with of the pitfalls to be found in Mathematics: the answer is right, but the method of solution, beautifully simple as it looks, is entirely wrong.

This can be most easily demonstrated by a *reductio ad absurdum*. Let the winning throw, for A and B alike, be 6. Then, by this method of solution, their chances are equal, since "the probability that B will have a throw after A is $\frac{31}{36}$"; which is also the probability "that A will throw again after B." Yet it is obvious that, as A begins, his "expectation" is better than B's.

The true solution will be best given, first, in the general form; and the formula, so obtained, can then be applied to the particular case.

Let A's chance of making his winning throw, each time he throws, be k; and similarly let B's chance be l.

Then A's chance of winning, in his first throw, is k; in his second, $(1-k).(1-l).k$; in his third, $(1-k)^2.(1-l)^2.k$; and so on for ever. Hence the limit, to which his "expectation" approaches, is the limit of

$$k.[1+(1-k).(1-l)+(1-k)^2.(1-l)^2+\&c.];$$

$$i.e., \quad k.\frac{1}{1-(1-k).(1-l)}; \quad i.e., \quad \frac{k}{k+l-kl}.$$

Similarly, B's chance of winning, in his first throw, is $(1-k).l$; in his second, $(1-k).(1-l).(1-k).l$; in his third, $(1-k)^2.(1-l)^2.(1-k).l$; and so on for ever. Hence his "expectation" approaches the limit of

$$1-k).l.[1+(1-k).(1-l)+(1-k)^2.(1-l)^2+\&c.]; \quad i.e. \ \frac{(1-k).l}{k+l-kl}.$$

Hence the ratio, of A's expectation to B's, is approximately $\frac{k}{(1-k).l}$.

In the given case, $k=\frac{5}{36}$, $l=\frac{6}{36}=\frac{1}{6}$; hence the required ratio $=\frac{30}{31}$. By a mere accident this happens to be the same as $\frac{1-l}{1-k}$, which accident has misled all the solvers into adopting this as a true formula.

In my "*reductio ad absurdum*" case, $k=l=\frac{5}{36}$; hence the required ratio $=\frac{36}{31}$.

It is worth noting that the ratio, $\frac{30}{31}$, is only *approximative*, the expectations of A and B being just *less* than the fractions $\frac{30}{61}$, $\frac{31}{61}$. If this were not so, the sum total of their expectations would equal 1; *i.e.*, it would be absolutely certain that one or other of them would win—whereas there is clearly a chance, though an indefinitely small one, that the game might go on for ever without either winning.

[Mr. SIMMONS remarks that the last portion of the above Note is "extremely unmathematical. A's expectation is represented with *perfect accuracy* by the series $\frac{5}{36}\left[1+\frac{155}{216}+\left(\frac{155}{216}\right)^2+\left(\frac{155}{216}\right)^3+\ldots\right]$, and it is erroneous to say that the sum of this series is only *approximately* equal to $\frac{30}{61}$. When we say that $a=b$ approximately, we mean that a and b differ by at least some *conceivable* quantity. Thus, we say rightly that the ratio of the circumference of a circle to its diameter is approximately equal to 3·14159265; but it would be wrong to say that it is approximately equal to $4\left(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots\right)$. The game may go on for ever without either A or B winning. True, but this is taken into account, and allowed for, by the above series going on for ever without stopping. Mr. Dodgson's reasoning, if it were correct, might be applied equally to almost every probability question. For instance, we might say that it is "worth noting" that, in the case of a triangle whose vertices are taken at random on the circumference of a given circle, the chance of its being acute-angled is only *approximately* $\frac{1}{4}$, and that of its being obtuse-angled only *approximately* $\frac{3}{4}$, '*because there is clearly a chance, though an indefinitely small one, that the triangle may be right-angled!*' Has not Mr. DODGSON, in his anxiety to avoid one of the aforesaid mathematical pit-falls, walked straight into another?"]

8006. (By Professor BYOMAKESA CHAKRAVARTI, M.A.)—If the temperature of an infinite solid have different uniform values V, V′ on opposite sides of a given plane, prove (1) that, at any subsequent time t, the temperature is given by the expression

$$\frac{V+V'}{2}+\frac{V-V'}{\sqrt{\pi}}\int_0^{\frac{x}{2\sqrt{kt}}} e^{-z^2}\,dz,$$

x being measured from the plane towards the side where the temperature is initially V; and (2), if the reasoning be applied to the case of the earth, supposed to have been cooling for 200,000,000 years from a uniform temperature, and if the numerical value of k be 400, when a foot is the unit of length and a year the unit of time, prove that, at any particular instant, at a depth of about 76 miles the rate of cooling is greatest; and at a depth of about 130 miles the rate of cooling has reached its maximum value at that place for all time.

Solution by Professor HAUGHTON, F.R.S.

The first part of this question was solved by Sir WILLIAM THOMSON (*Trans. Royal Society, Edinburgh*, 1862).

The second part is solved by

$$\frac{d^2v}{dt^2}=\phi\,(x^2-8kt)\times e^{-\frac{x^2}{4kt}}=0,$$

where ϕ is a function having no part in the question.

These two factors give $x_1 = 151{\cdot}51$ miles, $x_2 = 135{\cdot}52$ miles.

My first answer is double that of the PROPOSER.

N.B.—I have solved this problem in conformity with the time-honoured illusion of a cooling globe. My private opinion, however, is that the earth is mainly a globe of metallic iron, having a probable temperature of 460°F., in most parts; with occasional hot layers depending on greater collision between the meteorites out of which it was formed, and on local chemical actions depending chiefly on the oxidation of the iron.

The heat derived from the interior of the earth is very contemptible, only sufficient to melt a quarter of an inch of ice in the year; whereas the sun, in the same time, melts 150 feet of ice.

7818. (By MORGAN JENKINS, M.A.)—1. If on the three sides of a triangle ABC there be described any three similar triangles BDC, CEA, and AFB, either all externally or all internally, having their angles in the same order of rotation, and the angles which are contiguous to the same corner of the triangle ABC equal to each other, prove that the three straight lines AD, BE, and CF meet in a point O, which is also the common point of intersection of the circles BDC, CEA, and AFB.

2. If the homologous sides of these similar triangles be produced to meet, viz., FB and EC in D′, DC and FA in E′, and EA and DB in F′, the triangles BD′C, CE′A, and AF′B are also similar triangles having their angles in the same order of rotation, and equal angles contiguous to the same corner of the triangle ABC; hence the three circles circumscribing these similar triangles and the three straight lines AD′, BE′, CF′ meet in the same point O′.

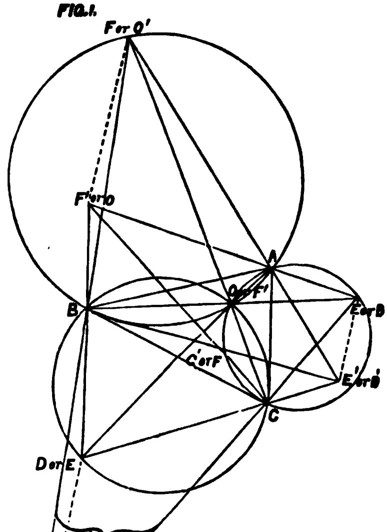
3. The straight lines DD′, EE′, FF′ are parallel to one another and to OO′.

4. O and O′ are confocal points with regard to the triangle ABC, that is, are the two foci of a central conic touching the sides of the triangle, or O′ may be determined by making the angles CBO′, CAO′ equal to the angles ABO, BAO respectively in opposite directions of rotation, and then angle BCO′ is equal to the angle ACO.

5. The sides of the triangle BCD′ or either of the other two similar triangles are proportional to the rectangles AO, BC; BO, CA; and CO, AB; and in like manner for the sides of the triangle BCD and the two similar triangles; that is, in the typical case, if lengths h, k, l meet at a point within a triangle and make angles θ, ϕ, and ψ with one another, then a triangle which has its angles equal to $\theta - A$, $\phi - B$, and $\psi - C$, will have its sides proportional to ah, bk, and cl.

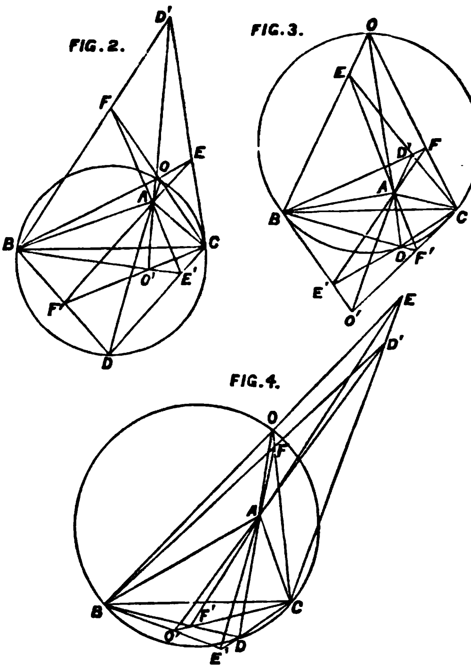
Solution by Proposer.

FIG. 1.



Let AD, BE meet in O, then, since the triangles ECA, BCD are similar, having equal angles ECA, BCD and the sides about those angles proportionals, therefore by the addition or subtraction of the angle ACB, according as the triangles ECA, BCD are described externally or internally to the triangle ABC, the angles ECB, ACD are equal, and, by alternation, the sides about those angles are proportionals; hence the triangles ECB, ACD are similar to each other; in like manner, the triangles DBA, CBF are similar to each other; and the triangles BAE, FAC are similar to each other. Now let θ', ϕ', ψ' denote the angles of the three similar triangles BDC, CEA, AFB; θ' being the angle which is opposite to A in the first triangle and adjacent to A in the other two, and similarly for ϕ' and ψ': also let θ, ϕ, and ψ be used for $\pi - \theta'$, $\pi - \phi'$, and $\pi - \psi'$ respectively.

First, taking the cases where the similar triangles are described externally, of the three sums $A+\theta'$, $B+\phi'$, and $C+\psi'$ at least two, say the last two, must be less than π, because the sum of all six angles is equal to 2π. Therefore AD lies within the angle BAC, and AO within the same angle or its vertical angle. The relations $B+\phi'<\pi$ and $C+\psi'$ are equivalent to $\phi>B$ and $\psi>C$: the necessary and sufficient condition that O may be within the triangle ABC is that we may also have $\theta>A$ or $A+\theta'<\pi$, as follows from Euclid I. 21.



In every case, when the similar triangles are applied externally, O must be on the opposite side of BC to D: if therefore $\theta<A$, then O must be in the vertical compartment lying within BA produced and CA produced; but of the pairs of points B, E and C, F, O may lie either between both pairs, outside both pairs, or between one pair B and E, and outside another pair C and F, as shown in figures 2, 3, and 4: but O is never in a base compartment, if the similar triangles are applied externally.

In Fig. 1, where $\theta>A$, $\phi>B$, and $\psi>C$, $\angle$ EBC $=$ $\angle$ ADC, that is, the $\angle$ OBC $=$ $\angle$ ODC, and B and D are on the same side of OC; therefore O is concyclic with B, D, and C. Also $\angle$ BEC $=$ $\angle$ DAC, that is, $\angle$ OEC $=$ $\angle$ OAC; therefore O is concyclic with C, E, and A. Therefore O is the intersection of two segments applied to BC and CA on the same side as the opposite vertices and containing $\angle$ BOC $=\theta$, $\angle$ COA $=\phi$; but $\theta+\phi+\psi=2\pi$, and BOC + COA + AOB $=2\pi$. Therefore the remaining $\angle$ AOB $=\psi$; and O is concylic with A, F, and B. Hence, since it may be proved in a similar manner that BE and CF meet on the other point of intersection of the circles CAE, BAF, the straight lines AD, BE, and CF meet in the point O: this proves theorem (1).

Again, the $\angle$ BCD' = supplement of BCE $=$ $\angle$s CBE + CEB $=$ $\angle$s CBO + CAO $=\psi-C$: similarly, $\angle$ D'BC $=\phi-B$, and therefore $\angle$ BD'C $=\theta-A$. Similarly for the other two triangles ACE' and BAF'; and this proves theorem (2).

Since $EC : CA = BC : CD$ and $CA : CE' = CD' : BC$, $\therefore$ $EC : CE' = CD' : CD$ and $\angle ECE'$ = vertical $\angle D'CD$, therefore the triangles ECE', $D'CD$ are similar, and EE' is parallel to DD'. Similarly, FF' is parallel to DD'. By transversals $AO : OD = EE' . FF' : DD' . (EE' + FF')$ and $AO' : O'D$ = the same ratio; $\therefore$ OO' is parallel to DD', and therefore also to EE' and FF': this proves theorem (3). Also $AO : OO' = AD : DD'$ and $OO' : OB = EE' : BE$; $\therefore$ $AO : OB = AD . EE' : BE . DD' = AC . EC : CE . CD' = AC : CD'$, and $\angle ACD' = \psi - C + C = \psi = \angle AOB$. Therefore the triangles AOB and ACD' are similar, and the $\angle CAO'$ is equal to the $\angle BAO$: in like manner for the other angles. This proves theorem (4). The same result may be obtained thus:—Make the $\angle CAD'$ = the $\angle BAO$ in the contrary direction of rotation, and make rectangle $AO . AD'$ = rectangle $AB . AC$, then the triangles BAO, $D'AC$ are similar: hence, the $\angle D'CA$ = the $\angle AOB$, and therefore the $\angle D'CB = \psi - C$: similarly, the $\angle D'BC = \phi - B$, and therefore the $\angle BD'C = \theta - A$. Similarly for the triangles CAE' and ABF', which are similar to the triangle BCD'. Therefore, by theorem (1), AD', BE', and CF' meet in a point O', and OO' is parallel to DD', because the rectangles $AO . AD'$ and $AO' . AD$ are each equal to the rectangle $AB . AC$. Since $AO : OB = AC : CD'$, $\therefore CD' = \dfrac{BO . AC}{AO}$: similarly, $BD' = \dfrac{CO . AB}{AO}$, and $BC = \dfrac{AO . BC}{AO}$; and this proves theorem (5).

These proofs hold good for the other figures, when suitable modifications are made in the relations of the angles θ', ϕ', ψ', and in the positions of the points. In figures 2, 3, and 4, θ is $< A$ and O is in the vertical compartment opposite to the angle BAC; but O' is in the base compartment opposite to BC, the similar triangles $BD'C$, &c. being applied to the sides of the triangle ABC internally. Fresh figures will not be required for the cases where the similar triangles are applied internally; for, in figure 1, the triangles ECB and ACD are similar, and are applied to BC and AC internally; and in like manner for other triangles. If, therefore, we change E into D, D into E, E′ into D′, and D′ into E′, O′ into F and O into F′, F′ into O and F into O′, we have a figure where both sets of similar triangles are applied internally. It may be noticed that in the typical case the three parallel straight lines DD', EE', FF' are all in the same direction; but in the other cases one of them is in the opposite direction to the other two.

The use of similar triangles as here applied was suggested by the Sylvester-Kempe extension of Hart's cell, where they are applied to the sides of a contra-parallelogram (vide *Nature*, July, 1875).

7812. (By Professor Genese, M.A.)—If CA, CB are semi-conjugate diameters of an ellipse, and P, Q two points on CA, CB produced such that $AP . BQ = 2CA . CB$, prove that BP, AQ intersect on the ellipse.

Solution by Professor Joseph Neuberg; *and the* Proposer.

Soit H un point de l'ellipse. Les droites AH, BQ engendrent des faisceaux homographiques, et déterminent sur CB, CA des divisions

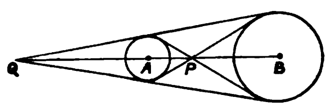
homographiques. Dans celles-ci, les points correspondant à l'infini sont B et A; donc le produit BQ . AP est constant. [A longer proof, by Algebra, is given in Vol. 43, p. 55.]

7947 & 7956. (By Âsûtosh Mukhopâdhyây, B.A., F.R.A.S.)—(7947.) Prove that the locus of points (H), from which tangents drawn to two given circles are in the ratio of their radii, is a circle passing through the centres of similitude as the extremities of a diameter.

(7956.) Prove that (1) the locus of points from which tangents drawn to two fixed circles are in *any* given ratio, is a circle; and (2) for all values of this ratio, the locus of the centre of this locus-circle is the straight line that joins the centres of similitude of the fixed circles.

Solution by Rev. J. L. Kitchin, M.A.; J. O'Regan; *and others.*

(7947.) Let A, B be the circles; P, Q the centres of similitude; $S_1 = 0$, $S_2 = 0$ the circles A and B: then $S_1 - r_1^2$, $S_2 - r_2^2$ are the squared tangents from H to the circle; hence we have $\dfrac{S_1 - r_1^2}{r_1^2} = \dfrac{S_2 - r_2^2}{r_2^2}$;



therefore $S_1 r_2^2 - S_2 r_1^2 = 0$ locus of H, which is obviously a circle. It clearly satisfies the points P and Q, and therefore passes through them, and it is symmetrical about PQ; therefore PQ is the diameter.

(7956.) This problem gives $S_1 - m^2 S_2 + m^2 r_2^2 - r_1^2 = 0$, and the locus of the centre is on AP.

8038. (By J. P. Johnstone, B.A.)—If a cone of the second degree, whose vertex moves on a right line, intersects a quadric in a pair of planes, one of which is fixed, the other developes a cone of the second degree having its vertex at the intersection of the polar line of the fixed line with the fixed plane.

Solution by Professor Malet, F.R.S.

Let the quadric be the sphere $x^2 + y^2 + z^2 + 2lx + 2my + 2nz + d = 0$, and the fixed plane the plane at infinity. Now, if the vertex of the cone lie on the axis of z, its equation will be $x^2 + y^2 + (z - \gamma)^2 = 0$, which intersects the sphere in the plane $2lx + 2my + 2z(n + \gamma) + d - \gamma^2 = 0$, and this plane, as γ varies, envelopes the cone $z^2 + 2lx + 2my + 2nz + d = 0$, the vertex of which is the intersection of the plane at infinity with the line $z = 0$, $lx + my + nz = 0$; but the polar line of the axis of z with respect to the sphere is $z + n = 0$, $lx + my + nz + d = 0$, which is parallel to the former line; and therefore the theorem is proved in this case, and, being projective, is true for any quadric.

8042 & 8078. (By Professor SYLVESTER, F.R.S.)—(8042.) Let A, B, C, D be the perpendiculars upon a plane from the points a, b, c, d, the angles of a pyramid whose volume is P. Required (1) to prove that

$$\Sigma (ab)^4 (C-D)^2 - 2\Sigma (ab)^2 (ac)^2 (D-B)(D-C) + 2\Sigma (ab)^2 (cd)^2 (A-C)(A-D) + (B-C)(B-D) = -144P^2.$$

Show also (2) how to find the Constant Homogeneous Quadratic Function of the five perpendiculars from five points in space of four dimensions upon any hyper-plane drawn thereon.

(8078.) If, in a system of quadruplanar coordinates, for which $x_1 + x_2 + x_3 + x_4$ expresses the plane at infinity, $A_1 A_2 A_3 A_4$ is the pyramid of reference; show that (1) $\Sigma (A_1 A_2)^2 xy$ is the sphere which circumscribes it; and hence (2) if p_1, p_2, p_3, p_4 are the perpendicular distances of A_1, A_2, A_3, A_4 from any variable plane, the following determinant is a constant, and find its value:—

$$\begin{array}{cccccc} . & p_1 & p_2 & p_3 & p_4 & . \\ p_1 & . & (A_1A_2)^2 & (A_1A_3)^2 & (A_1A_4)^2 & 1 \\ p_2 & (A_2A_1)^2 & . & (A_2A_3)^2 & (A_2A_4)^2 & 1 \\ p_3 & (A_3A_1)^2 & (A_3A_2)^2 & . & (A_3A_4)^2 & 1 \\ p_4 & (A_4A_1)^2 & (A_4A_2)^2 & (A_4A_3)^2 & . & 1 \\ . & 1 & 1 & 1 & 1 & . \end{array}$$

Solution by Professor NEUBERG.

Voici la solution que nous avons donnée de ces questions dans un mémoire publié en 1869 (*Etudes sur les Coordonnées Tétraédriques*).

Soient h_1, h_2, h_3, h_4 les hauteurs du tétraèdre de référence. Cherchons d'abord la distance XY = δ de deux points dont les *coordonnées barycentriques* sont (x_1, x_2, x_3, x_4), $(y_1, \ldots)$; par exemple, x_1 est le quotient de la distance de X au plan $A_2 A_3 A_4$, divisée par h_1. Si dans la relation

$$\begin{vmatrix} 1 & \cos \delta h_1 & \cos \delta h_2 & \cos \delta h_3 \\ \cos h_1 \delta & 1 & \cos h_1 h_2 & \cos h_1 h_3 \\ \cos h_2 \delta & \cos h_2 h_1 & 1 & \cos h_2 h_3 \\ \cos h_3 \delta & \cos h_3 h_1 & \cos h_3 h_2 & 1 \end{vmatrix} = 0, \quad \text{on fait} \quad \cos \delta h_1 = \frac{h_1 (x_1 - y_1)}{\delta}, \ldots,$$

on voit que δ^2 est une fonction du second degré des différences $x_1 - y_1$, $x_2 - y_2$, $x_3 - y_3$. Mais on peut éliminer les carrés $(x_1-y_1)^2$, $(x_2-y_2)^2$, $(x_3-y_3)^2$ en remarquant que $\quad \Sigma x_1 = 1, \; \Sigma y_1 = 1, \; \Sigma (x_1 - y_1) = 0,$

d'où $(x_1-y_1)^2 = -(x_1-y_1)(x_2-y_2) - (x_1-y_1)(x_3-y_3) - (x_1-y_1)(x_4-y_4)$, etc. Par conséquent, on peut poser

$$\delta^2 = d_{12}(x_1-y_1)(x_2-y_2) + \ldots + d_{34}(x_3-y_3)(x_4-y_4) \ldots\ldots\ldots\ldots (1).$$

Pour trouver les coefficients inconnus d_{12}, ..., nous ferons coïncider XY successivement avec chacune des six arêtes $A_1 A_2$, $A_1 A_3$, ..., ce qui donne

$$d_{12} = -(A_1 A_2)^2, \ldots, \quad d_{34} = -(A_3 A_4)^2.$$

Soient Y le centre de la sphère $A_1 A_2 A_3 A_4$, R le rayon, X un point quelconque de la sphère. L'équation de celle-ci sera

$$\Sigma d_{12}(x_1-y_1)(x_2-y_2) = R^2,$$

ou $$\Sigma d_{12} x_1 x_2 + \Sigma d_{12} y_1 y_2 - \Sigma d_{12}(x_1 y_2 + x_2 y_1) = R^2 \ldots\ldots\ldots\ldots (2).$$

Les inconnues y_1, y_2, y_3, y_4, R résultant des equations

$$\left.\begin{aligned}\Sigma d_{12}y_1y_2-(d_{12}y_2+d_{13}y_3+d_{14}y_4) &= R^2\\ \Sigma d_{12}y_1y_2-(d_{21}y_1+d_{23}y_3+d_{24}y_4) &= R^2\\ \Sigma d_{12}y_1y_2-(d_{31}y_1+d_{32}y_2+d_{34}y_4) &= R^2\\ \Sigma d_{12}y_1y_2-(d_{41}y_1+d_{42}y_2+d_{43}y_3) &= R^2\end{aligned}\right\}\ \ldots\ldots\ldots\ldots\ldots\ldots(3),$$

qui expriment que la sphère passe par les points (1, 0, 0, 0), etc. ; de plus

$$y_1+y_2+y_3+y_4 = 1\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

Retranchons de (2) les équations (3) multipliées respectivement par x_1, x_2, x_3, x_4 ; nous aurons $\Sigma d_{12}x_1x_2 = 0$, ce qui démontre la première partie de la question.

Si on fait la somme des équations (3) multipliées par y_1, y_2, y_3, y_4, on trouve $\Sigma d_{12}y_1y_2 = -R^2$, ce qui réduit les équations (3) à

$$d_{12}y_2+d_{13}y_3+d_{14}y_4 = -R^2, \text{ etc.}\ \ldots\ldots\ldots\ldots\ldots\ldots(3'),$$

Soit E le déterminant

$$\begin{matrix} . & d_{12} & d_{13} & d_{14} & 1\\ d_{21} & . & d_{23} & d_{24} & 1\\ d_{31} & d_{32} & . & d_{34} & 1\\ d_{41} & d_{42} & d_{43} & . & 1\\ 1 & 1 & 1 & 1 & . \end{matrix}$$

qui, comme on sait, est égal à $288\,(A_1A_2A_3A_4)^2$; et désignons par $E_{11}, E_{12}, \ldots$ les mineurs de E. Les équations (3′) et (4) donnent

$$R^2 = -\frac{E_{55}}{E},\quad y_1 = -\frac{E_{51}}{E}, \text{ etc.}$$

Soient de nouveau X, Y deux points quelconques de l'espace. On a évidemment

$$x_1-y_1 = \delta\frac{\cos\delta h_1}{h_1} = \delta\lambda_1, \ldots$$

$\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \frac{1}{\lambda_3}, \frac{1}{\lambda_4}$ étant les longueurs des droites A_1B_1, A_2B_2, A_3B_3, A_4B_4 parallèles à XY et terminées aux faces du tétraèdre de référence. Les quantités λ vérifient les identités

$$\lambda_1+\lambda_2+\lambda_3+\lambda_4 = 0,\quad \Sigma d_{12}\lambda_1\lambda_2 = 1\ \ldots\ldots\ldots\ldots(5, 6),$$

qui se tirent de (1) et de $\Sigma(x_1-y_1) = 0$; on peut les appeler *coefficients de direction* des droites parallèles à XY et les considérer comme *coordonnées* du point à l'infini sur ces droites.

Soit V l'angle de deux droites XY, YZ dont les coefficients de direction sont $(\lambda_1, \ldots)$, $(\mu_1, \ldots)$. Si XY = XZ = 1, on a

$$(YZ)^2 = (XY)^2+(XZ)^2-2XY.XZ.\cos V = 2-2\cos V;$$

mais $\quad x_1-y_1 = \lambda_1,\quad x_1-z_1 = \mu_1,\quad y_1-z_1 = \mu_1-\lambda_1$, etc.,

par suite $\quad (YZ)^2 = \Sigma d_{12}(\mu_1-\lambda_1)(\mu_2-\lambda_2) = 2-2\cos V.$

D'où, à cause de (6), $\quad \cos V = \frac{1}{2}\Sigma d_{12}(\mu_1\lambda_2+\mu_2\lambda_1).$

La condition de perpendicularité de deux directions est donc

$$\Sigma d_{12}(\mu_1\lambda_2+\mu_2\lambda_1) = 0\ \ldots\ldots\ldots\ldots\ldots\ldots(7).$$

Elle pourrait encore être obtenue en exprimant que les points à l'infini sur les directions rectangulaires sont conjugués harmoniques par rapport à la sphère $A_1A_2A_3A_4$ dont l'équation est $\Sigma d_{12}x_1x_2 = 0$.

Un plan quelconque P est le lieu des droites XY menées par un point fixe Y et perpendiculaires à une direction fixe ($\lambda_1, \ldots$). Son équation peut donc se déduire de (7) en faisant

$$\mu_1 = \frac{x_1 - y_1}{\delta}, \quad \mu_2 = \frac{x_2 - y_2}{\delta}, \ldots,$$

ce qui donne, après avoir posé $\phi(x) = \Sigma d_{12} x_1 x_2$ et représenté par $\phi_1(x)$, $\phi_2(x), \ldots$ les demi-dérivées partielles de $\phi(x)$ par rapport à $x_1, x_2, \ldots$,

$$\Sigma (x_1 - y_1)\phi_1(\lambda) = 0.$$

Si l'on fait encore $\Sigma y_1 \phi_1(\lambda) = K = K\Sigma y_1$, l'équation de P devient

$$[\phi_1(\lambda) - K] x_1 + \ldots = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8).$$

Soit maintenant $XZ = \delta$ la perpendiculaire abaissée d'un point quelconque X sur P ; nous aurons $x_1 - z_1 = \delta\lambda_1$, $z_1 = x_1 - \delta\lambda_1$, etc. Exprimons que le point Z est dans le plan (8), en tenant compte de (5) et (6) ; il vient

$$\delta = [\phi_1(\lambda) - K] x_1 + \ldots \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(9).$$

Si $K = 0$, le plan P passe par le centre de la sphère $A_1A_2A_3A_4$; car les coordonnées de ce centre vérifient les équations (3′) ou

$$\phi_1(x) = \phi_2(x) = \phi_3(x) = \phi_4(x),$$

et l'on a identiquement $\Sigma x_1 \phi_1(\lambda) = \Sigma \lambda_1 \phi_1(x)$. D'ailleurs le plan polaire du point à l'infini ($\lambda_1, \ldots$) par rapport à la sphère $\phi(x) = 0$ a précisément pour équation $\Sigma x_1 \phi_1(\lambda) = 0$.

En général, K est la distance du centre de la sphère $A_1A_2A_3A_4$ au plan (8).

D'après la formule (9), si p_1, p_2, p_3, p_4 sont les distances de P à A_1 (1, 0, 0, 0), A_2 (0, 1, 0, 0), etc., on a

$$p_1 = \phi_1(\lambda) - K, \quad p_2 = \phi_2(\lambda) - K, \quad p_3 = \phi_3(\lambda) - K, \quad p_4 = \phi_4(\lambda) - K \ldots(10).$$

On a aussi $\qquad \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 0 \quad \ldots\ldots\ldots\ldots\ldots(5),$

et, en faisant la somme des équations (10) multipliées par $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ et réduisant au moyen de (6) $p_1\lambda_1 + p_2\lambda_2 + p_3\lambda_3 + p_4\lambda_4 = 1 \ldots\ldots\ldots\ldots\ldots\ldots(11),$

Eliminons $\lambda_1, \lambda_2, \lambda_3, \lambda_4$, K entre (11), (10) et (5) ; nous aurons

$$\begin{vmatrix} 1 & p_1 & p_2 & p_3 & p_4 & 0 \\ p_1 & . & \frac{1}{2} d_{12} & \frac{1}{2} d_{13} & \frac{1}{2} d_{14} & 1 \\ p_2 & \frac{1}{2} d_{21} & . & \frac{1}{2} d_{23} & \frac{1}{2} d_{24} & 1 \\ p_3 & \frac{1}{2} d_{31} & \frac{1}{2} d_{32} & . & \frac{1}{2} d_{34} & 1 \\ p_4 & \frac{1}{2} d_{41} & \frac{1}{2} d_{42} & \frac{1}{2} d_{43} & . & 1 \\ . & 1 & 1 & 1 & 1 & 0 \end{vmatrix} = 0.$$

On conclut de là que le déterminant de M. Sylvester a pour valeur $\frac{1}{2}E$ ou $-144\,(A_1A_2A_3A_4)^2$.

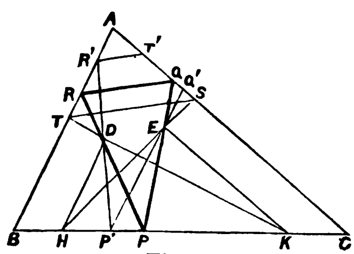
Si l'on développe ce déterminant suivant les produits des déterminants partiels formés avec les colonnes extrêmes d'une part et les lignes extrêmes d'autre part, on obtient la formule de la question 8042.

La relation (11) est susceptible de la généralisation suivante : *Des parallèles menées pas les sommets d'un tétraèdre* $A_1A_2A_3A_4$ *rencontrent les faces opposées aux points* B_1, B_2, B_3, B_4 *et un plan quelconque aux points* C_1, C_2, C_3, C_4 ; *on a la relation*

$$\frac{A_1C_1}{A_1B_1} + \frac{A_2C_2}{A_2B_2} + \frac{A_3C_3}{A_3B_3} + \frac{A_4C_4}{A_4B_4} = 1.$$

8012. (By the Editor.)—From any point P in the base BC of a triangle ABC, lines PDR, PEQ are drawn through fixed points D, E to meet AB, AC in R, Q. Draw DH, EK respectively parallel to AB, AC, meeting the base in H, K, and produce HE, KD to meet AC, AB respectively in S, T; then prove that (1) ΔAQR is a *maximum* when QR is parallel to ST; (2) for other positions of P the rectangle SQ . TR is constant; (3) hence, or otherwise, give an easy construction for finding the position (P_m) of P for the maximum triangle AQR; (4) prove also that ΔAQR is a *minimum* when QR is parallel to ST (the corresponding position of P being denoted by p_m); (5) the positions of QR in (1) and (4) are equidistant from ST and on opposite sides of it; (6) the range HP_mKp_m is harmonic; (7) if $[HPP_mK] = [KP'P_mH]$, the areas of the triangles AQR corresponding to the points P, P′ are equal; (8) for all positions of P, SQ varies as the ratio of HP to PK; (9) the ratio of AR to AQ depends only on the anharmonic ratio of $KeP'P$, where P′ is determined as in (7) and e is the intersection of AE with BC; (10) hence, or otherwise, find the relation between the two positions of P corresponding to two *parallel* positions of QR; and (11) express the *ratio* of any two values of the area of AQR in terms of the corresponding positions of P.

I. *Solution by the* Rev. T. C. Simmons, M.A.



(Fig. 1.)

1. Let P be such that, when PDR, PEQ are drawn, RQ is parallel to TS. Through any other point P′ draw the lines P′DR′, P′EQ′; also R′r′ parallel to TS. Then, if we imagine other points Q″Q‴..., r″r‴... taken in AC in like manner, it is evident that the two systems QQ′Q″Q‴..., Qr′r″r‴... will be homographic, Q being a double point. Moreover, the point S considered as a member of the first system will correspond with ∞ in the second, and the same point considered as a member of the second system will correspond with ∞ in the first. Hence the two systems will form an involution having S for centre. (See Townsend's *Modern Geometry*, Vol. II., p. 292, &c.) That is to say, $SQ'.Sr' = SQ^2$ = constant. Now the arithmetic mean between two lines exceeds the geometric mean, so that $Qr' > QQ'$; and from this it follows, on the same principle, that $AQ'.Ar' < AQ^2$. But $AR'.AQ = Ar'.AR$. Therefore $AQ'.AR' < AQ.AR$. That is, AQR is the required maximum triangle.

2. Since TR′ varies as Sr′, it follows at once, from above, that $SQ'.TR' = SQ.TR$ = constant.

3. We will give three independent constructions for determining the position of QR:

(α) Through any point P′ draw P′EQ′, P′DR′, R′r′, as in Fig. 1, and take $SQ^2 = SQ'.Sr'$. Produce QE to meet BC in P, and PD to meet AB in R. This perhaps is the simplest construction.

(β) Take Q′Q″Q‴ any three positions of Q, and $r'r''r'''$ the three corresponding positions of r; the lines R′r′, R″r″, R‴r‴ being in this case

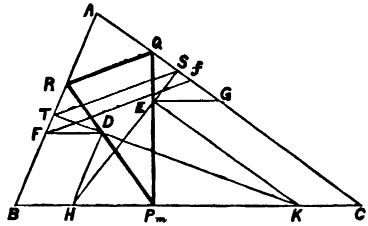
parallel to *any* fixed direction. On $Q'r''$ describe a semicircle, and on $Q''r'$ a semi-ellipse, the squares of whose axes, perpendicular and parallel to AC, are in the ratio $Q'Q'''.r''r''' : Q''Q'''.r'r'''$. From their point of intersection draw a perpendicular to AC; this will meet it in the required point Q. (Compare CASEY's *Sequel to Euclid*, p. 136.) This construction, it will be seen, does not involve the points S and T.

(γ) Determining H, K, S, T as before, draw DF, EG parallel to BC as in the figure (Fig. 2); also Ff parallel to TS. Then $[HP_mK\infty] = [SQ\infty G]$ on account of the common vertex E. That is,

$$\frac{HP_m \,.\, K\infty}{P_mK \,.\, H\infty} = \frac{SQ \,.\, \infty G}{Q\infty \,.\, SG},$$

or

$$\frac{HP_m}{P_mK} = \frac{SQ}{SG}.$$



(Fig. 2.)

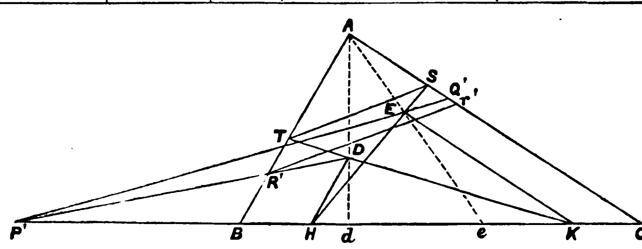
Again, on account of the common vertex D, and the parallels RQ, TS, Ff, we have $[HP_mK\infty] = [\infty QSf]$, whence

$$\frac{HP_m \,.\, K\infty}{P_mK \,.\, H\infty} = \frac{\infty Q \,.\, Sf}{QS \,.\, \infty f} \quad \text{or} \quad \frac{HP_m}{P_mK} = \frac{Sf}{QS}.$$

Combining these two results, we get $HP_m^2 : P_mK^2 :: Sf : SG$. This is an extremely simple method for determining the position of P_m.

4. In Fig. 3, let AD, AE be produced to meet BC in d, e respectively. Then we have, corresponding to different positions of P, these values for ΔAQR:

Positions of P	$-\infty$ to H	H to d	d to P_m, then to e	e to K	K to $+\infty$
area of AQR	finite to ∞	∞ to 0	0 to maximum, then to 0	0 to ∞	∞ to finite



(Fig. 3.)

so that there will be a minimum value of ΔAQR for some position of P, either to the left of H, or to the right of K. Take P' in the former region, and draw P'R'D, P'EQ'; also $R'r'$ parallel to TS (Fig. 3). Then, as in (1), it will be seen that the points such as Q', r' taken on SC form an involution lying now on the other side of S and again having S for centre. Whence, as before, $SQ'.Sr' =$ constant. Take $Sq_m^2 = SQ'.Sr'$; then, if a semicircle be described on $Q'r'$, and tangents drawn thereto from S and

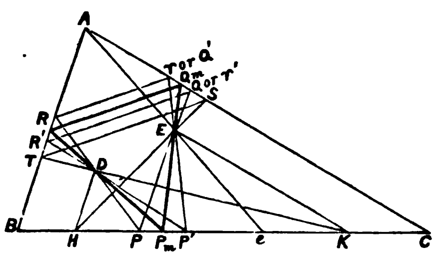
A, it is easily seen that the geometric mean between AQ′ and Ar' has its extremity farther from A than the geometric mean between SQ′ and Sr', except when Q′ and r' coincide. Whence AQ′. Ar' is a minimum at the point q_m, and, as in (1), we get for the minimum triangle that which has its base parallel to ST. It is hardly necessary to refer to the apparent paradox, that in the above figure the *minimum* triangle is greater than the *maximum*.

5. Moreover, as the latter involution belongs to the same system as the former, it may be inferred that $Sq_m = SQ_m$; or, if this proof be deemed unsatisfactory,

6. We can deduce, as in (γ) above, that p_m being the position of P for this other triangle whose base is parallel to ST, $Hp_m^2 : p_mK^2 :: Sf : SG$, so that $Hp_m : p_mK :: HP_m : P_mK$, showing that $[HP_mKp_m]$ is harmonic. Whence, since $[HP_mKp_m] = [SQ_m\infty\, q_m]$, it follows that $SQ_m = Sq_m$; an independent method for deducing (5).

7. Through any point P draw the lines PEQ, PDR, Rr, as in Fig. 4. Then, since SQ . Sr always $= SQ_m^2$, it is evident that, if we produce rE to P′, and P′D to R′, then draw R′r' parallel to TS, r' will coincide with Q. Hence this will be the case of the equality of the triangles ARQ, AR′Q′. Also

$[KP'P_mH] = [\infty\, Q'Q_mS]$



(Fig. 4.)

or $[\infty\, r\, Q_mS]$, which $= [SQQ_m\infty]$ on account of the involution, which latter range again $= [HPP_mK]$. Hence theorem (7) is proved.

8. In Fig. 4 we have $[SQQ_m\infty] = [HPP_mK]$ or

$$\frac{SQ\,.\,Q_m\infty}{SQ_m\,.\,Q\infty} = \frac{HP\,.\,P_mK}{HP_m\,.\,PK} \quad \text{or} \quad SQ = \frac{SQ_m\,.\,P_mK}{HP_m}\,.\,\frac{HP}{PK};$$

which, the points Q_m and P_m being fixed, shows that SQ varies as HP/PK.

9. Referring again to Fig. 4, we see that $[ArQ\infty] = [eP'PK]$ or $\dfrac{Ar\,.\,Q\infty}{AQ\,.\,r\infty} = [eP'PK]$. Hence $\dfrac{AR}{AQ}$, which varies as $\dfrac{Ar}{AQ}$, depends only on $[eP'PK]$.

10. As P moves along BC from $-\infty$ to $+\infty$, it will be seen that RQ revolves completely through the four quadrants, and comes twice into a position of parallelism with any given direction.

If P_1, P_2 denote the corresponding positions of P, we see, from (9), that $[eP_1P_1'K] = [eP_2P_2'K]$. This relation enables us to connect the position of P_2 with that of P_1. For, P_1 being given, P_1' can be determined from (7); hence $[eP_1P_1'K]$ is known. Hence, for the determination of P_2, we have the two equations $[eP_2P_2'K]$ = constant, and $[HP_2P_mK] = [KP_2'P_mH]$; from which, by the elimination of P_2', we can find P_2.

11. Let P′, P″ denote any two positions of P; and AQ′R′, AQ″R″ the corresponding triangles. Then, d being taken as in Fig. 3,

$$\frac{\Delta AQ'R'}{\Delta AQ''R''}=\frac{AQ'.AR'}{AQ''.AR''}=\frac{AQ'.Q''\infty}{AQ''.Q'\infty}\cdot\frac{AR'.R''\infty}{AR''.R'\infty}$$
$$=\frac{eP'.P''K}{eP''.P'K}\cdot\frac{dP'.P''H}{dP''.P'H}=\frac{eP'.P'd}{HP'.P'K}\div\frac{eP''.P''d}{HP''.P''K}.$$

This gives the ratio of the two triangles corresponding to the points P′, P″. It will be at once seen that the absolute area of any triangle AQ′R′ depends only on the ratio $eP'.P'd : HP'.P'K$, and hence can be given at once in terms of the maximum triangle, and the position of P′. It can also be seen that there will be *two* positions of P′ corresponding to any given magnitude of area of $\Delta AQ'R'$, as in theorem (7) proved above.

II. *Solution of Parts* (1) *and* (3) *by* D. Biddle.

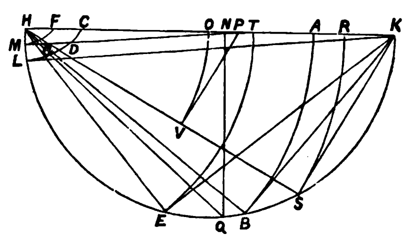
1. Draw DF, EG parallel to BC (Fig. 2). Then, let HK = 1, DH = d, EK = e, AF = f, AG = g, DF = m, EG = n, and HP = x.

Then $x : d = m : f-AR$, and $1-x : e = n : g-AQ$; whence
$$AR=(fx-dm)/x,\quad\text{and}\quad AQ=[g(1-x)-en]/(1-x).$$
Now, in order that the triangle ARQ may be a maximum, AR . AQ must be a maximum also. But $AR.AQ=fg-\frac{efn}{1-x}-\frac{dgm}{x}+\frac{demn}{x(1-x)}$. Therefore, since fg is the limit, $\frac{efn\,x+dgm\,(1-x)-demn}{x(1-x)}$ = a minimum, and
$$[dm\,(g-en)+(efn-dgm)\,x]\,/\,x\,(1-x)$$
$$=[dm\,(g-en)+(efn-dgm)\,x+(efn-dgm)\,h]\,/\,[x\,(1-x)+h-2hx-h^2];$$
whence $(efn-dgm)\,x^2+2\,(dgm-demn)\,x=dgm-demn$. But $1:d=m:TF$ and $1:e=n:SG$. Moreover, $f-TF$ = AT, and $g-SG$ = AS.

Let TF = p, SG = q, AT = t, AS = s. Then $dm=p$, and $en=q$, and the above equation becomes $(fq-gp)\,x^2+2\,(gp-pq)\,x=gp-pq$ and
$$x=\pm\left[\left(\frac{gp-pq}{fq-gp}\right)^2+\frac{gp-pq}{fq-gp}\right]^{\frac12}-\frac{gp-pq}{fq-gp}=(sp)^{\frac12}\,/\,[(sp)^{\frac12}+(qt)^{\frac12}].$$

For example, let HK = 77, DH = 44, EK = 26, AF = 60, AG = 68, DF = 23, EG = 23. Then
$$p=44.23/77^2,\quad q=26.23/77^2,\quad t=(60.77-44.23)/77^2,$$
$$s=(68.77-26.23)/77^2,$$
and x = ·596 . HK = 45·892 as in the diagram. We have, also

$$RT+p:m=d:x,$$
$$\therefore\quad RT=p\,(1-x)/x,$$
and, assuming that QR is parallel to ST,
$$\frac{s}{t}RT+q:n=e:1-x,$$
therefore
$$RT=qx\Big/\frac{s}{t}(1-x)$$
$$=p\,(1-x)/x,$$
whence $x=(sp)^{\frac12}/[(sp)^{\frac12}+(qt)^{\frac12}]$, as before.



(Fig. 5.)

3. The foregoing analysis also enables us to find the point P, and to construct the triangle PQR, *geometrically*. For we have

$$x = 1 \Big/ \left\{ 1 + \left(\frac{qt}{sp}\right)^{\frac{1}{2}} \right\} = \tfrac{1}{2} \Big/ \left\{ \tfrac{1}{2} + \tfrac{1}{2} \left(\frac{qt}{sp}\right)^{\frac{1}{2}} \right\}.$$

Accordingly, upon HK, with centre O (Fig. 2), describe a semi-circle, and mark off HA $= s$, HT $= t$, HC $= p$, and HF $= q$. Describe the arcs AB, TE, and draw CD, FG parallel to KB, KE respectively. Then HD $= sp$ and HG $= qt$. Further, describe the arc DL and join LK, also make HM (on HL) = HG, and draw MN parallel to LK; then HN $= qt / sp$. Draw NQ at right angles to HK and join HQ; then

$$\text{HQ} = \left(\frac{qt}{sp}\right)^{\frac{1}{2}}.$$

Finally, make OR $= \frac{1}{2}$HQ, describe the arcs RS, OV, join SK and draw VP parallel to SK. Then $\text{HP} = \frac{1}{2} \Big/ \left\{ \frac{1}{2} + \frac{1}{2} \left(\frac{qt}{sp}\right)^{\frac{1}{2}} \right\}$, and P is the required vertex of the triangle PQR.

[We have $\dfrac{\text{QC}}{\text{EK}} = \dfrac{\text{PC}}{\text{PK}}, \; \dfrac{\text{SC}}{\text{EK}} = \dfrac{\text{HC}}{\text{HK}}$; therefore $\dfrac{\text{QS}}{\text{EK}} = \dfrac{\text{HP}\,.\,\text{KC}}{\text{PK}\,.\,\text{HK}}$;

similarly, we find $\dfrac{\text{RT}}{\text{DH}} = \dfrac{\text{PK}\,.\,\text{BH}}{\text{HP}\,.\,\text{HK}}$;

hence we obtain

$$\text{QS}\,.\,\text{RT} = \frac{\text{BH}\,.\,\text{HD}\,.\,\text{CK}\,.\,\text{KE}}{\text{HK}^2} = \frac{\text{SC}\,.\,\text{TB}\,.\,\text{BH}\,.\,\text{CK}}{\text{BK}\,.\,\text{CH}} = \text{a constant.}$$

Now ΔAQR will be a maximum when the *constant* magnitude (AS . AT + QS . RT) diminished by the *variable* magnitude AQ . AR is a minimum, that is to say, when AT . QS + AS . RT is a minimum. And it is well known that, if QS, RT be (as in this problem) variable lines whose rectangle QS . RT is constant, then (AS, AT being constant lines) AT . QS + AS . RT will be a maximum or a minimum when AT . QS = AS . RT, that is to say, when AS : QS = AT : RT, that is, when QR is parallel to ST.]

8008. (By Professor Wolstenholme, M.A., Sc.D.)—Two conics are met by a transversal in the points P, Q; P′, Q′ respectively, and AA′ is a common tangent; the straight lines AP, AQ meet the straight lines A′P′, A′Q′ in four points; prove that these four points and the four common points of the two conics lie on one conic.

Solution by R. Lachlan, B.A.; Dr. Curtis; *and others.*

Let U, V be any two conics, and let two other conics u, v be drawn through the points in which U and V are cut by any conic S; then we have $u \equiv \text{S} + \lambda\text{U}$, $v \equiv \text{S} + \mu\text{V}$; whence $u - v \equiv \lambda\text{U} - \mu\text{V}$, or the four points

common to u and v, and the four points common to U and V, lie on a conic. The theorem in the question is easily deduced from this.

[On pp. 24, 25 of Vol. XLIII. (Quest. 7842) it is shown that, if two circles are met by a transversal in the points P, Q; P′, Q′, respectively, and AA′ be a common tangent touching the circles in A, A′, respectively, the four points in which the straight lines AP, AQ meet the straight lines A′P′, A′Q′, lie on a circle having a common radical axis with the given circles. If in this theorem the three circles be projected into conics, and the line at infinity into a line intersecting them in two common points, the theorem here enunciated results.]

7998. (By F. PURSER, M.A., and Professor HAUGHTON, F.R.S.)—Four points on a quartic lie on a line (A); three other points lie on a line (B); three other points lie on a line (C); there are (of course) two other real points, lying on (B) and (C) respectively: prove that, for every possible quartic passing through the above ten points, the line joining the remaining two real points passes through a fixed point which can be constructed.

Solution by JAMES R. HOLT, B.A.

1. Taking the three lines as sides of triangle of reference, let the equation of the quartic be Q = 0, and let the results of putting $x = 0$, $y = 0$, $z = 0$ respectively in Q be X = 0, Y = 0, Z = 0. These three equations give the points in which the quartic meets the three lines; hence all the roots of X = 0, three of the roots of Y = 0, and three of the roots of Z = 0 are fixed. Let the coefficients of x^4, y^4, z^4 be a, b, c. Let the factors of X = 0 be $y+\alpha_1 z$, $y+\alpha_2 z$, &c.; let the factors of Y = 0 be $z+\beta_1 x$, $z+\beta_2 x$, &c., and the factors of Z = 0 be $x+\gamma_1 y$, $x+\gamma_2 y$, &c.; then

$$\alpha_1\alpha_2\alpha_3\alpha_4 = \frac{c}{b};\quad \beta_1\beta_2\beta_3\beta_4 = \frac{a}{c};\quad \gamma_1\gamma_2\gamma_3\gamma_4 = \frac{b}{a};$$

hence $\alpha_1\alpha_2\alpha_3\alpha_4\beta_1\beta_2\beta_3\beta_4\gamma_1\gamma_2\gamma_3\gamma_4 = 1$. But in this all the quantities except β_4 and γ_4 are fixed; hence $\beta_4\gamma_4$ is fixed. Let the line passing through the points (β_4, γ_4) be $lx+my+nz = 0$.

Then $\dfrac{l}{n} = \beta_4$, $\dfrac{m}{l} = \gamma_4$. Therefore $\dfrac{m}{n}$ is fixed, and therefore the line cuts $x = 0$ in a fixed point.

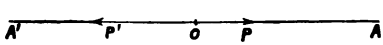
2. To construct the point if the line joining β_1, γ_1 meets $x = 0$ in $y+\alpha_1' z = 0$, we have $\beta_1\gamma_1\alpha_1' = 1$. Similarly for the other pairs of points. Hence $\alpha_1\alpha_2\alpha_3\alpha_4 = \alpha_1'\alpha_2'\alpha_3'\alpha_4'$. But $\dfrac{\alpha_1}{\alpha_1'}$, $\dfrac{\alpha_2}{\alpha_2'}$, $\dfrac{\alpha_3}{\alpha_3'}$ are known anharmonic

ratios. Compounding them (which can be done by a linear construction), we know $\frac{a_4}{a_4'}$, and therefore can determine the required point by a linear construction.

[Mr. Purser thinks that perhaps the most convenient method of constructing the fixed point which lies, as Mr. Holt has shown by a process the same as his own, on the line (A) is the following:—Let the four points on (A) be denoted by P, Q, P′, Q′; the three on (B) by S, T, U; and the three on (C) by S′, T′, U′. Then one quartic through the ten points is the pair of conics PQS′T′U, P′Q′STU′, and for this quartic the line $lx + my + nz = 0$ is that joining the remaining intersections V, V′ of these conics with (B), (C) respectively. We have only then to construct the points V, V′, and the intersection of VV′ with (A) will be the fixed point required. Now V, V′ are given immediately by applying Pascal's theorem to the hexagons PQS′T′UV, P′Q′STU′V′. We have thus the following construction:—Join the intersection of the lines PS′, QU to the intersection of (B), (C), and let the joining line meet QT′ in M. Then PM meets (B) in V. A precisely similar construction determines V′. It appears from above theorem that, if all the four points on (B) be given, the fourth point on (C) is determined, being of course found by joining the fourth given point on (B) to the fixed point on A, determined as above, and producing this line to meet (C).]

Proofs of the Formulæ $s = \frac{1}{2}ft^2$, &c. By J. Walmsley, B.A.

1. Assume OA to represent the path of a particle P moving from rest at O during time t with uniform acceleration f. At A the velocity accumulated will be ft.



In AO produced make OA′ = OA = s suppose. Then A′A = $2s$.

Suppose a particle P′ to move from O along OA′, starting with velocity ft, but with a uniform *retardation* f which will reduce ft to zero in t seconds. Clearly the motion of P′ during this time will be that of P reversed, and P′ will come to rest at A′ as P is arriving at A.

Consider the *relative* velocity of P from P′. This is simply the rate of increase of the distance which separates them; and

= vel. of P′ from O + vel. of P from O = a *uniform* velocity ft;

since, at any instant, P has just gained in velocity what P′ has lost.

Hence, by formula for uniform motion,

$$2s = ft \cdot t = ft^2; \quad \text{therefore } s = \tfrac{1}{2}ft^2.$$

Taking initial velocities as u and $u + ft$, we get, in this way, $s = ut + \frac{1}{2}ft^2$.

We will obtain this result as an illustration of another method.

2. Suppose we impose on a particle two *accelerated* velocities in one straight line, namely u with acceleration f, and $u+ft$ with acceleration $-f$. In time t the former will increase to $u+ft$, and the latter decrease to u.

By the Second Law of Motion, the space described in the same time will be the sum of the spaces due to the component velocities themselves. Call it $2s$.

But the resultant velocity is uniform, namely $2u+ft$, for one component is gaining at the same rate as the other is losing,

therefore $$2s = (2u+ft)\,t = 2ut + \tfrac{1}{2}ft^2.$$

Again, the space due to each component is the same, for the motion due to one of them is precisely the same as that due to the other taken the reverse way. It follows that the space due to each velocity is s.

We conclude, therefore, that for a particle moving along a straight line, with initial velocity u and acceleration f, $s = ut + \frac{1}{2}ft^2$. The same method will, of course, apply when $u = 0$.

The remaining formulæ $s = \frac{1}{2}vt$, &c. easily follow as usual.

8016. (By T. Muir, LL.D.)—Show that, if Σa stands for $a+b+c+d$, the persymmetric determinant

$$\begin{vmatrix} 1, & \tfrac{1}{4}\Sigma a, & \tfrac{1}{6}\Sigma ab \\ \tfrac{1}{4}\Sigma a, & \tfrac{1}{6}\Sigma ab, & \tfrac{1}{4}\Sigma abc \\ \tfrac{1}{6}\Sigma ab, & \tfrac{1}{4}\Sigma abc, & abcd \end{vmatrix} = \tfrac{1}{2}\left[\tfrac{1}{6}\Sigma ab - \tfrac{1}{2}(ab+cd)\right]\left[\tfrac{1}{6}\Sigma ab - \tfrac{1}{2}(ac+bd)\right] \times \left[\tfrac{1}{6}\Sigma ab - \tfrac{1}{2}(ad+bc)\right].$$

Solution by B. Hanumanta Rau, M.A.; J. O'Regan; *and others.*

$$1728\Delta = -8(\Sigma ab)^3 + 36\Sigma ab\,(8abcd + \Sigma a\,.\,\Sigma abc) - 108\left[(\Sigma abc)^2 + abcd(\Sigma a)^2\right],$$

$$\text{or } 432\Delta = (\Sigma ab)^3 - 3(\Sigma ab)^2(\Sigma ab) + 9(\Sigma ab)\left[\Sigma a\,.\,\Sigma abc - 4abcd\right]$$
$$-27\left[(\Sigma abc)^2 + abcd\,(\Sigma a)^2 - 4abcd\,\Sigma ab\right]$$
$$= \left[\Sigma ab - 3(ab+cd)\right]\left[\Sigma ab - 3(ac+bd)\right]\left[\Sigma ab - 3(ad+bc)\right].$$

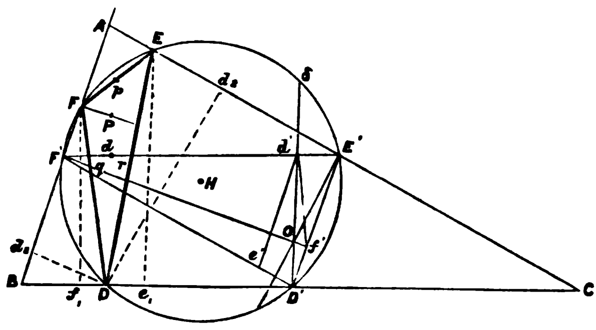
For $$\Sigma a\,.\,\Sigma abc - 4abcd = (ab+cd)(ac+bd) + (\quad)(\quad) + (\quad)(\quad),$$

and $$(\Sigma abc)^2 + abcd\,(\Sigma a)^2 - 4abcd\,.\,\Sigma ab = \Sigma\,(a^2b^2c^2) + abcd\,.\,\Sigma\,(a^2)$$
$$= (ab+cd)(ac+bd)(ad+bc).$$

7938. (By R. Tucker, M.A.)—ABC is a triangle of which DEF, D′E′F′ (D, D′ on BC, &c.) are the pedal and medial triangles respectively; prove that the six Simson-lines, taken from each vertex with

reference to the other triangle, the circum-circle being the nine-point circle of ABC, pass through a point on the line mentioned in Quest. 7900, and is the centre of Mr. H. M. Taylor's circle.

Solution by Charlotte A. Scott, B.Sc.



Let perpendicular from D′ on E′F′ meet circle again in δ. Then we know that Dδ and the Simson-line of D with respect to DEF make equal angles with D′δ. But D′δ is a diameter; therefore Simson-line of D is parallel to D′H, *i.e.*, perpendicular to EF.

Let d, d_2, d_3 be feet of perpendiculars from D on E′F′, CA, AB. Then, since APD, AFd_3, AEd_2 are respectively collinear, d_2d_3 is parallel to EF; therefore the Simson-line of D goes through d, and is perpendicular to d_2d_3.

But d is centre of circle Ad_2Dd_3; therefore Simson-lines of D, E, F bisect d_2, d_3, e_3, e_1, f_1, f_2 at right angles; and therefore, since d_2, d_3, e_3, e_1, f_1, f_2 lie on circle, centre Q, the three Simson-lines of D, E, F pass through Q.

Again, D′D is perpendicular to DP, which is bisector of angle EDF; therefore line joining feet of perpendiculars from D′ on FD, ED is perpendicular to DD′; *i.e.*, Simson-line of D′ with regard to DEF is perpendicular to DD′. Moreover, since D′H is perpendicular to FE and bisects it, the Simson-line goes through bisection of EF.

Now e_1, f_1 are points on circle, centre Q; therefore Q lies on line perpendicular to e_1f_1 at its bisection, *i.e.*, Q lies on line through bisection of EF, perpendicular to DD′, *i.e.*, Q lies on Simson-line of D′ with regard to DEF, and therefore the three Simson-lines of D′E′F′ with regard to D, E, F pass through Q.

Let pqr be medial triangle of DEF, $d'e'f'$ pedal triangle of D′E′F′. Then pqr, $d'e'f'$ are both similar to DEF, and of half the linear dimensions; therefore they are equal. Also they are similarly situated, and therefore pd', qe', rf' are parallel, as are all lines joining corresponding points. Now O is in-centre to $d'e'f'$, and Q is in-centre to pqr [since pQ is parallel to DP, &c.], therefore OQ is parallel to $d'p'$, &c., and therefore OQ is parallel to line joining centroids of pqr, $d'e'f'$.

Now (centroid of $d'e'f'$ to centroid D′E′F′) is parallel to (centroid of

DEF to centroid ABC), *i.e*, to (centroid pqr to centroid D′E′F′), therefore centroids of D′E′F′, $d'e'f'$, pqr are colliuear. Therefore OQ is parallel to line joining centroids of D′E′F′, DEF, *i.e.*, OQ is parallel to OK, and therefore Q lies on line mentioned in 7900.

7969. (By Professor SÁRADÁRANJAN RÁY, M.A. Extension of Question 7865.)—On the sides of any triangle, similar and *similarly situated* polygons are described, and equal masses are placed at all the corners; prove that the centre of gravity of the masses coincides with that of the triangle.

Solution by DR. CURTIS; A. GORDON, M.A.; *and others.*

Let p_1, p_2, p_3 denote the three altitudes of the given triangle, let n be the number of sides in each of the polygons, G_1, G_2, G_3 the centres of gravity of a system of masses, each $= m$, placed at *each* angle of *each* of the polygons on the sides a, b, c respectively, thereby introducing masses $2m$ at the points A, B, C. If from G_1, G_2, G_3 perpendiculars be let fall on a, b, c, they will themselves be proportional to these sides, and may be denoted by λa, λb, λc, and will divide a, b, c, respectively, in the same ratio, say $\kappa : 1-\kappa$. The centre of gravity of this entire system will be the centre of gravity of three masses, each $= nm$, placed at G_1, G_2, G_3, and, if $\bar{z}$ be its distance from c,

$$3nm\bar{z} = nm\left[\kappa a \sin B + \lambda a \cos B + (1-\kappa)\, b \sin A + \lambda b \cos A - \lambda c\right],$$

or, as $\qquad c = a \cos B + b \cos A$, and $a \sin B = b \sin A$,

$$3\bar{z} = b \sin A = p_1, \text{ therefore } \bar{z} = \tfrac{1}{3}p_1; \quad \text{similarly} \quad \bar{y} = \tfrac{1}{3}p_2, \; \bar{x} = \tfrac{1}{3}p_1;$$

therefore the centre of gravity of the system coincides with that of the given triangle. Again, as the centre of gravity of three masses, each $= m$, placed at A, B, C, coincides with the centre of gravity of the triangle, these three masses may be left out of account, thus reducing the system above considered to that cons sting of a mass m placed at each angle of the figure constructed as in the Question; therefore, &c.

7922. (By Professor SYLVESTER, F.R.S.)—Prove that the equation in quaternions $x^2 - px = 0$ has four roots, and that these roots, if regarded as belonging to the *square* of the equation, obey HARRIOT's law.

Solution by Professor MATHEWS, B.A.

Let $\qquad p = \delta + ai + \beta j + \gamma k, \quad x = \omega + \xi i + \eta j + \zeta k,$

where i, j, k are rectangular unit vectors; then

$$x^2 = \omega^2-\xi^2-\eta^2-\zeta^2+2\omega\xi i+2\omega\eta j+2\omega\zeta k,$$

$$px = \delta\omega-\alpha\xi-\beta\eta-\gamma\zeta+(\alpha\omega+\delta\xi-\gamma\eta+\beta\zeta)\,i+(\beta\omega+\delta\eta-\alpha\zeta+\gamma\xi)\,j$$
$$+(\gamma\omega+\delta\zeta-\beta\xi+\delta\eta)\,k.$$

Hence, if $x^2 = px$,

$$\left.\begin{array}{l}\omega^2-\xi^2-\eta^2-\zeta^2 = \delta\omega-\alpha\xi-\beta\eta-\gamma\zeta,\ 2\omega\xi = \alpha\omega+\delta\xi-\gamma\eta+\beta\zeta \\ 2\omega\eta = \beta\omega+\delta\eta-\alpha\zeta+\gamma\xi,\quad 2\omega\zeta = \gamma\omega+\delta\zeta-\beta\xi+\delta\eta\end{array}\right\}\ \ldots(A).$$

The last three equations may be written

$$(2\omega-\delta)\,\xi+\gamma\eta-\beta\zeta = \alpha\omega,\qquad -\gamma\xi+(2\omega-\delta)\,\eta+\alpha\zeta = \beta\omega,$$

$$\beta\xi-\alpha\eta+(2\omega-\delta)\,\zeta = \gamma\omega.$$

Let
$$\Delta = \begin{vmatrix} 2\omega-\delta & \gamma & -\beta \\ -\gamma & 2\omega-\delta & \alpha \\ \beta & -\alpha & 2\omega-\delta \end{vmatrix}$$

$$= (2\omega-\delta)\,[4\omega^2-4\omega\delta+\delta^2+\alpha^2]+\gamma\,(\alpha\beta+2\omega\gamma-\gamma\delta)-\beta\,(\gamma\alpha-2\omega\beta+\beta\delta)$$
$$= (2\omega-\delta)\,[4\omega^2-4\omega\delta+\alpha^2+\beta^2+\gamma^2+\delta^2],$$

then $\Delta\,.\,\xi = \omega\,[\alpha\,(4\omega^2-4\omega\delta+\delta^2+\alpha^2)+\beta\,(\alpha\beta-2\gamma\omega+\gamma\delta)+\gamma\,(\gamma\alpha-\beta\delta+2\beta\omega)]$

$$= \omega\alpha\,[4\omega^2-4\omega\delta+\alpha^2+\beta^2+\gamma^2+\delta^2];$$

hence
$$4\omega^2-4\omega\delta+\alpha^2+\beta^2+\gamma^2+\delta^2 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(B),$$

or else
$$\xi = \frac{\omega\alpha}{2\omega-\delta},\quad \eta = \frac{\omega\beta}{2\omega-\delta},\quad \zeta = \frac{\omega\gamma}{2\omega-\delta}.$$

Substituting in the first of equations (A), we get, after transposing,

$$\omega\,(\omega-\delta) = (\alpha^2+\beta^2+\gamma^2)\left\{\frac{\omega^2}{(2\omega-\delta)^2}-\frac{\omega}{2\omega-\delta}\right\} = -\frac{(\alpha^2+\beta^2+\gamma^2)\,\omega\,(\omega-\delta)}{(2\omega-\delta)^2};$$

hence $\omega = 0$, or δ, or else

$$(2\omega-\delta)^2 = -(\alpha^2+\beta^2+\gamma^2),\quad \omega = \tfrac{1}{2}\,[\delta\pm(-1)^{\frac{1}{2}}\,(\alpha^2+\beta^2+\gamma^2)^{\frac{1}{2}}],$$

the same values as would be derived from (B).

In the last expression, $(-1)^{\frac{1}{2}}$ must be taken to mean an imaginary *scalar* quantity, while $(\alpha^2+\beta^2+\gamma^2)^{\frac{1}{2}}$ means the arithmetical square root of $\alpha^2+\beta^2+\gamma^2$. Using quaternion notation, the values of ω or $\mathrm{S}x$ are

$$0,\ \mathrm{S}p,\ \tfrac{1}{2}\,[\mathrm{S}p\pm(-1)^{\frac{1}{2}}\,.\,\mathrm{TV}p],$$

while
$$\mathrm{V}x = \xi i+\eta j+\zeta k = \frac{\omega}{2\omega-\delta}\,(\alpha i+\beta j+\gamma k) = \frac{\omega}{2\omega-\delta}\,\mathrm{V}p$$

$$= 0,\ \mathrm{V}p,\ \frac{\mathrm{S}p\pm(-1)^{\frac{1}{2}}\,\mathrm{TV}p}{2\,[\pm(-1)^{\frac{1}{2}}\,\mathrm{TV}p]}\,.\,\mathrm{V}p = 0,\ \mathrm{V}p,\ \tfrac{1}{2}\,[\mathrm{TV}p\mp(-1)^{\frac{1}{2}}\,.\,\mathrm{S}p]\,\mathrm{UV}p.$$

Thus, finally, the four values of x are

$$x_1 = 0,\qquad x_2 = p,\qquad x_3 = \tfrac{1}{2}p+\tfrac{1}{2}\,(-1)^{\frac{1}{2}}\,[\mathrm{TV}p-\mathrm{S}p\,.\,\mathrm{UV}p],$$

$$x_4 = \tfrac{1}{2}p-\tfrac{1}{2}\,(-1)^{\frac{1}{2}}\,[\mathrm{TV}p-\mathrm{S}p\,.\,\mathrm{UV}p].$$

Hence $$x_1+x_2+x_3+x_4 = 2p,$$
$$x_1x_2+\ldots = p^2+\tfrac{1}{4}\,[p^2+(\mathrm{TV}p-\mathrm{S}p\,.\,\mathrm{UV}p)^2]$$
$$= p^2+\tfrac{1}{4}\,[p^2-(\mathrm{V}p)^2-(\mathrm{S}p)^2-2\mathrm{S}p\,.\,\mathrm{V}p] = p^2,$$
$$x_1x_2x_3+\ldots = x_2x_3x_4 = \tfrac{1}{4}p\,[p^2+(\mathrm{TV}p-\mathrm{S}p\,.\,\mathrm{UV}p)^2] = 0, \quad x_1x_2x_3x_4 = 0,$$
the same relations as for the roots of $x^4-2px^3+p^2x^2=0$, *i.e.*, of
$$(x^2-px)^2 = 0.$$

8045. (By Professor WOLSTENHOLME, M.A., Sc.D.)—Through each point P of a given straight line is drawn a straight line making a given angle with the polar of P with respect to a given conic; prove that (1) the envelope of such straight line is in general a parabola, but degenerates into a point when the given angle is that which the given straight line makes with the diameter of the given conic conjugate to it; and (2) this point is the focus of any parabolic envelope.

Solution by SAMUEL ROBERTS, M.A.; A. MUKHOPÂDHYÂY, B.A.; *and others.*

Take for the given conic the equation
$$(a,\,b,\,c,\,f,\,g,\,h\between x,\,y,\,1)^2 = 0,$$
for the given line $x=0$, and for the generating line $y-mx-n=0$. Then, if t be the tangent of the given angle, we have
$$m+\frac{nh+g}{nb+f} = t\left(n-m\,\frac{nh+g}{nb+f}\right),$$
and, substituting, $y-mx$ for n, we get
$$m^2(th+b)\,x-m\,[(tb-h)\,x+(th+b)\,y+tg+f]+(tb-h)\,y+tf-g = 0\ldots(a).$$
The envelope is evidently a parabola.

When $m=i$ (i for $\pm\sqrt{-1}$), (a) becomes
$$(i-t)\,(hx-by-f)-(1+it)(bx+hy+g) = 0,$$
and the focus is determined by equating the coefficients of $(i-t)$ and $(1+it)$ to zero. But the tangent of the angle which the axis of y makes with the conjugate diameter in question, is $-\dfrac{b}{h}$, which, substituted for t in (a), causes the coefficient of m^2 to vanish, leaving a system of straight lines passing through the focus.

7719. (By ASÛTOSH MUKHOPÂDHYÂY.)—Show that, if
$$\frac{bx+ay-cz}{a^2+b^2} = \frac{cy+bz-ax}{b^2+c^2} = \frac{az+cx-by}{c^2+a^2},$$
then (1) $$x\,(a^2-bc)+y\,(b^2-ac)+z\,(c^2-ab) = 0,$$

implying $$\frac{x+y+z}{a+b+c} = \frac{ax+by+cz}{ab+bc+ca};$$

and (2) $$(ab+bc+ca)\left(x^{\frac12}+y^{\frac12}+z^{\frac12}\right)\left(y^{\frac12}+z^{\frac12}-x^{\frac12}\right)\left(z^{\frac12}+x^{\frac12}-y^{\frac12}\right)\left(x^{\frac12}+y^{\frac12}-z^{\frac12}\right)$$
$$= (ax+by+cz)^2.$$

Solution by B. Hanumanta Rau, M.A.; Sarah Marks; *and others.*

1. Putting each of the expressions equal to k and solving for x, y, and z, we obtain $x = k(b+c)$, $y = k(c+a)$, and $z = k(a+b)$, therefore
$$x(a^2-bc)+y(b^2-ac)+z(c^2-ab)$$
$$= k\left[a^2(b+c)+b^2(c+a)+c^2(a+b)-bc(b+c)-ca(c+a)-ab(a+b)\right] = 0,$$
or $$a^2x+b^2y+c^2z = bcx+cay+abz.$$
Adding $(ca+ab)x+(bc+ab)y+(ca+bc)z$ to both sides, this becomes
$$(ab+bc+ca)(x+y+z) = (a+b+c)(ax+by+cz).$$
The second part of (1) also follows from the fact that each fraction $= 2k$.

2. $$(ax+by+cz)^2-(ab+bc+ca)(2yz+2zx+2xy-x^2-y^2-z^2)$$
$$= (a+b)(a+c)x^2+\ldots+\ldots-2a(b+c)yz-\ldots-\ldots$$
$$= \frac{1}{k^2}\left[x^2yz+y^2zx+z^2xy-2kaxyz-2kbxyz-2kcxyz\right]$$
$$= \frac{xyz}{k}\left[x+y+z-2k(a+b+c)\right] = 0,$$
therefore $$(ax+by+cz)^2$$
$$= (ab+bc+ca)\left(x^{\frac12}+y^{\frac12}+z^{\frac12}\right)\left(y^{\frac12}+z^{\frac12}-x^{\frac12}\right)\left(z^{\frac12}+x^{\frac12}-y^{\frac12}\right)\left(x^{\frac12}+y^{\frac12}-z^{\frac12}\right).$$

7935. (By G. Heppel, M.A.)—Three lines, no two of which are parallel, are given by their equations. Express the condition that the origin may be within the triangle formed by them.

Solutions by (1) Rev. T. C. Simmons, M.A.; (2) *the* Proposer.

1. Write the equations in the form $a_1x+b_1y=1$, $a_2x+b_2y=1$, $a_3x+b_3y=1$, and denote the angles the lines make with the axis of x by α, β, γ.

Now it will be seen, from figures drawn in various positions, that the necessary and sufficient condition that the origin should lie within the triangle is that $\sin(\alpha-\beta)$, $\sin(\beta-\gamma)$, $\sin(\gamma-\alpha)$ should be all of the same sign. Whence, substituting $\cos\alpha = \dfrac{a_1}{(a_1^2+b_1^2)^{\frac12}}$, $\sin\alpha = \dfrac{b_1}{(a_1^2+b_1^2)^{\frac12}}$, &c., we obtain, as the required condition, that $a_1b_2-b_1a_2$, $a_2b_3-b_2a_3$, $a_3b_1-b_3a_1$ must be all of the same sign.

2. If the origin is within, the angular points are all on the origin side of the opposite sides. They cannot be all on the non-origin side. But, in order that the intersection of (1) and (2) should be on origin side of (3), the sign of

$$\begin{vmatrix} a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \\ a_3 & b_3 & 1 \end{vmatrix} \div \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

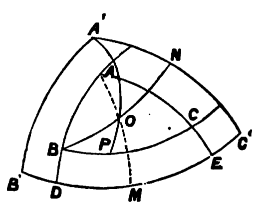
must be negative. The first determinate is symmetrical, therefore

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}, \quad \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}, \quad \begin{vmatrix} a_3 & b_3 \\ a_1 & b_1 \end{vmatrix}$$

must have the same sign.

8068. (By W. J. C. Sharp, M.A.)—Show that the angular radii of the circles inscribed in a spherical triangle and its associated triangles, are the complements of those of the circles described about the polar triangle and its associated triangles, and that the circles are consecutive.

Solution by Professor Neuberg.



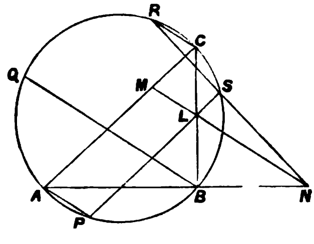
Soient ABC, A′B′C′ deux triangles polaires, C′ et B′ étant les pôles de AB, AC. Les arcs C′D, B′E sont égaux à un quadrant, d'où B′D = C′E. Le milieu M de DE est aussi le milieu de B′C′, et l'arc AM est à la fois bissecteur de l'angle BAC et perpendiculaire au milieu de B′C′. Par conséquent, le cercle inscrit à ABC et le cercle circonscrit à A′B′C′ ont même pôle O. L'arc A′O, qui passe par le pôle A′ de BC, est perpendiculaire sur BC; donc le rayon A′O du cercle circonscrit à A′B′C′ est complémentaire du rayon OP du cercle inscrit à ABC.

Le théorème du No. 8068 est fort connu, de même que le suivant :—*Les deux triangles* ABC, A′B′C′ *ont même orthocentre, et les hauteurs correspondantes sont supplémentaires.*

6871. (By J. L. McKenzie, B.A.)—The three sides BC, CA, AB of a triangle are cut by a straight line in L, M, N; and lines drawn through A, B, and C, parallel to LMN, cut the circumscribing circle of the triangle ABC in P, Q, and R; prove that the lines PL, QM, RN all cut the circle ABC in the same point.

Solution by the Rev. T. C. SIMMONS, M.A.

Let NR meet the circle in S and join LS; then ∠SNL = alternate angle SRC = ∠SBC, therefore a circle goes round SLBN; therefore ∠BSL = ∠BNL = alternate angle BAP, therefore SL produced passes through P. Similarly SM produced passes through Q.



[Mr. SIMMONS sends this solution of the Question because he finds a difficulty in understanding the one given on p. 66 of Vol. 40.]

8028. (By IRIS.)—Given two circles and a point O; draw a line PQ cutting the circles in P and Q respectively, so that the triangle OPQ may be similar to a given triangle ABC.

Solution by Dr. CURTIS; Professor CHAKRAVARTI, M.A.; *and others.*

If of a triangle of given species, OPQ, one angular point, O, is fixed, while P moves along a fixed straight line, or the circumference of a fixed circle, it is well known that the locus of the third angular point, Q, is a circle, denoted, for the purpose of reference, by A; if then Q be supposed to be restricted to a locus, B, which may be a straight line, a circle, or any fixed curve in the plane of the triangle, the point Q will be determined by the intersections of A and B, and the corresponding triangles obtained.

[For another solution, see Vol. XXIV., p. 112.]

4266. (By Professor SYLVESTER, F.R.S.)—If, by a mediate between two curves in respect to any point, be understood the curve which everywhere bisects each segment of any ray passing through that point intercepted between the two curves; prove that (1) every unicursal quartic having two nodes at infinity is a portion of the mediate of two similar conics placed with their axes parallel, in respect to a point situated on one of the conics, and that there always exist two real pairs of such conics (coinciding only in particular cases) of which any given quartic is a part-mediate; and (2) show also how to construct any unicursal quartic whatever by means of two general conics, a fixed point in either of them, and any one of their chords of (real or imaginary) intersection.

Solution by W. J. C. SHARP, M.A.

If $ax^2+by^2+2fx+2gy=0$ and $ax^2+by^2+2f'y+2g'x+c=0$ be the equations to the two conics referred to axes parallel to their axes of figure, the origin being at a point on the first, and if R and R′ be the radii vectores to these, which make an angle θ with the axis of x, R and R′ are determined by the equations

$$(a\cos^2\theta+b\sin^2\theta)\,\mathrm{R}+2\,(f\sin\theta+g\cos\theta)=0\ldots\ldots\ldots\ldots\ldots(1),$$

and $$(a\cos^2\theta+b\sin^2\theta)\,\mathrm{R}'^2+2\,(f'\sin\theta+g'\cos\theta)\,\mathrm{R}'+c=0\ldots\ldots\ldots(2),$$

whilst the corresponding radii vectores of the mediate are $\rho=\frac{1}{2}\mathrm{R}'$ or $\rho=\frac{1}{2}(\mathrm{R}+\mathrm{R}')$, values which correspond to two distinct portions of the mediate. Considering only the latter part, and eliminating R and R′ between $2\rho=\mathrm{R}+\mathrm{R}'$ and the equations (1) and (2), we have

$$[\rho\,(a\cos^2\theta+b\sin^2\theta)+f\sin\theta+g\cos\theta]$$

$$\times\,[\rho(a\cos^2\theta+b\sin^2\theta)+(f+f')\sin\theta+(g+g')\cos\theta]+c(a\cos^2\theta+b\sin^2\theta)=0.$$

The locus of the extremity of ρ is

$$[ax^2+by^2+fy+gx]\,[ax^2+by^2+(f+f')\,y+(g+g')\,x]+c\,(ax^2+by^2)=0\ldots(3),$$

a unicursal quartic with one node at the origin and two at infinity (at the points at infinity on the asymptotes to the given conics); so that the nodal triangle is formed by the parallels to the asymptotes through the origin and the line at infinity.

If $ax^2+by^2=0$ be the equation to the two sides not at infinity of the nodal triangle of a unicursal quartic with two nodes at infinity and one at the origin, the equation to the quartic may be reduced to

$$(ax^2+by^2)\,(ax^2+by^2+2hx+2ky)+lx^2+2mxy+ny^2=0,$$

which compared with (3) gives $2h=2g+g'$, $2k=2f+f'$, $l=g\,(g+g')+ca$, $2m=f(g+g')+g\,(f+f')$, $n=f\,(f+f')+cb$, which will give real values of $f, f+f'$, g and $g+g'$, if $h^2>h$ and $k^2>n$. Now the equation to the quartic may be written $(ax^2+by^2+hx+ky)^2=(h^2-l)\,x^2+2\,(hk-m)\,xy+(k^2-n)\,y^2$, which for a real quartic involves $h^2>l$ and $k^2>n$, and there will in general be two pairs of real conics, and these pairs coincide if

$$b\,(h^2-l)=a\,(k^2-n),$$

and the origin is the centre of (2).

The construction required in Part (2) of the question may be deduced by projecting infinity into z, one of the sides of the nodal triangle, and the axes of x and y into the other two. The conics become

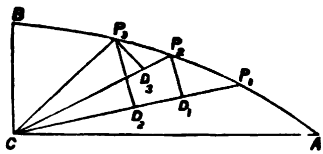
$$ax^2+by^2+2fyz+2gzx=0\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1^*),$$

$$ax^2+by^2+2f'yz+2g'zx+cz^2=0\ \ldots\ldots\ldots\ldots\ldots\ldots(2^*).$$

So that $x=0$, $y=0$ is in (1) and $z=0$ a chord of intersection. The projection of any point on the part-mediate is the point on the line through $x=0$, $y=0$, and that point which is harmonically conjugate to the second intersection of the line with (1*), one of its intersections with (2*) and the point where it meets $z=0$. The part-mediate is then projected into a unicursal quartic, having $x=0$, $y=0$, and $z=0$ for the sides of the nodal triangle.

7254. (By Professor MATZ, M.A.)—Given the axes $CA = 2a$ and $CB = 2b$ of an elliptic quadrant $AP_1P_2P_3B$; also the $\angle ACP_1 = \omega = 30°$, $\angle P_1CP_2 = \phi = 15°$, $\angle P_2CP_3 = \theta = 30°$: find (1) D_1P_2, D_2P_3, CD_1, CD_2, where P_2D_1, P_3D_2 are perpendicular to CP_1; also (2) these values for $a = b = 1$, $\omega = 0$.

Solution by R. KNOWLES, B.A.; Professor ROY, M.A.; *and others.*



1. From the equations

$y = \tan(\phi+\omega)\,x$, CP_2,

$y = \tan(\phi+\omega+\theta)\,x$, CP_3,

$b^2x^2 + a^2y^2 = 4a^2b^2$ the curve,

which belong, respectively, to CP_2, CP_3, and the curve, we obtain

$$CP_2 = \frac{2ab\sec(\phi+\omega)}{[b^2+a^2\tan^2(\phi+\omega)]^{\frac{1}{2}}} = \frac{2\sqrt{2}\,ab}{(a^2+b^2)^{\frac{1}{2}}},$$

$$CP_3 = \frac{2ab\sec(\phi+\omega+\theta)}{[b^2+a^2\tan^2(\phi+\omega+\theta)]^{\frac{1}{2}}} = \frac{4ab}{[(2+\sqrt{3})\,a^2+(2-\sqrt{3})\,b^2]^{\frac{1}{2}}}.$$

If $\omega = 30°$, $\phi = 15°$, $\theta = 30°$; and, from the triangles CD_1P_2 and CD_2P_3,

$$D_1P_2 = \frac{ab}{(a^2+b^2)^{\frac{1}{2}}}(\sqrt{3}-1), \quad CD_1 = \frac{ab}{(a^2+b^2)^{\frac{1}{2}}}(\sqrt{3}+1),$$

$$D_2P_3 = \frac{2^{\frac{3}{2}}\,ab}{[(2+\sqrt{3})\,a^2+(2-\sqrt{3})\,b^2]^{\frac{1}{2}}} = CD_2.$$

2. If $a = 1 = b$, $\omega = 0$, these become respectively

$$\frac{1}{\sqrt{2}}(\sqrt{3}-1); \quad \frac{1}{\sqrt{2}}(\sqrt{3}+1) \text{ and } \sqrt{2}.$$

8044. (By Professor HAUGHTON, F.R.S.)—The mean distance of Mars from the Sun is 121 millions of miles, and his periodic time is 687 days; calculate the mass of Mars (as compared with the Sun) from the following data as to the distance and periodic times of his two satellites—

	No. 1.	No. 2.
Distance......................	12483 miles.	6000 miles.
Periodic time...............	$30^h\ 14^m$.	$7^h\ 38^m$.

Solutions by (1) ALICE G. HUXHAM; (2) ADELAIDE HALL; (3) ÂSÛTOSH MUKHOPÂDHYÂY, B.A., F.R.A.S.

1. Taking distance and period of Deimos (satellite I.) as units, the distance and period of Mars are 9693 and 545. Hence mass of Mars / mass of Sun $= 545 \times 545 / (9693)^3 = 1/3066067$. Treating Phobos similarly,

the mass of Mars = 1 / 1757728. Thus the results are still discordant. If the elements given in BALL's *Elements of Astronomy* be taken (distance of Mars 141 millions, and of Deimos 14500 miles, with slightly altered periods) the results are 1 / 3106969 and 1 / 3093507, which are fairly accordant with each other and with Professor ASAPH HALL's calculation.

2. We use the formula $m = d^3 / t^2$, where m is mass of central body, and d, t the mean distance and periodic time of the planet or satellite. The results, by use of logarithms, are easily found to be:—From satellite I., 1 / 3062213; from satellite II., 1 / 1757900. The elements in BALL's *Astronomy* differ considerably from those given by Professor HAUGHTON, and give the mass of Mars; I., 1 / 3039690; II., 1 / 3027520.

3. Let R be the distance of Mars from the Sun, and T his periodic time; then, the "centrifugal force" of Mars in his orbit is $4\pi^2 R / T^2$. But, since Mars is retained in his path by the attraction of the Sun, which is proportional to the Sun's mass, and inversely as the square of the distance, we therefore have

$$\frac{\text{Mass of the Sun}}{R^2} = 4\pi^2 . \frac{R}{T^2},$$

whence $$\text{Mass of the Sun} = 4\pi^2 . \frac{R^3}{T^2} \text{ .... (1).}$$

Exactly for the same reason, if r and t denote the distance and periodic time of one of the satellites of Mars, its "centrifugal force" will be $4\pi^2 r / t^2$, which must be equal to the attraction of Mars, or

$$\frac{\text{Mass of Mars}}{r^2} = 4\pi^2 . \frac{r}{t^2}, \quad \text{or Mass of Mars} = 4\pi^2 . \frac{r^3}{t^2} \text{ .... (2).}$$

Hence we obtain $$\mu = \frac{\text{Mass of the Sun}}{\text{Mass of Mars}} = \left(\frac{R}{r}\right)^3 \left(\frac{t}{T}\right)^2.$$

Therefore $$\log \mu = 3 (\log R - \log r) - 2 (\log T - \log t).$$

Let us first take the first satellite. Here

$$R = 141 \times 10^6, \qquad T = 687 \times 24 \times 60',$$
$$r = 12483, \qquad t = 1814'.$$
$$\log R = 8{\cdot}1492191, \qquad \log T = 5{\cdot}9953192,$$
$$\log r = 4{\cdot}0963190, \qquad \log t = 3{\cdot}2586374.$$

Hence $$\log \mu = 6{\cdot}6853367, \qquad \mu = 4845478.$$

If we take the second satellite, we have $r_1 = 6000$, $t_1 = 458'$;

$$\log r_1 = 3{\cdot}7781513,$$
$$\log t_1 = 2{\cdot}6608655,$$

which give $$\log \mu_1 = 6{\cdot}4442960, \qquad \mu_1 = 2781608.$$

This great discrepancy between the two results is principally due to the fact that the elements of the first satellite as given in the question are too small. If we put $r = 14600$, as in LOOMIS, p. 223, we have $\log r = 4{\cdot}16435$,

whence $$\log \mu = 6{\cdot}4812350, \quad \text{and} \quad \mu = 3028551.$$

Of these two values, that obtained from the second satellite is the nearer to the value given in Dr. HAUGHTON'S *Astronomy*, p. 22, Table I. If we take the mean of the two values, we have

$$2 \log \mu = 12 \cdot 9255310,$$

$$\log \mu = 6 \cdot 4627655, \quad \text{whence} \quad \mu = 2900453,$$

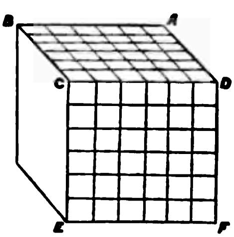
which is greater than the more accurate value given in the table, where $\mu = 2680337$. It is also to be remembered that the discrepancy arises partly from neglecting the ellipticity of the orbits. If we take the mass of the Sun to be unity, we have, from the respective sources affixed, the following five values, arranged in order of accuracy, for the mass of Mars :—

(1) ·0000003730874 (HAUGHTON'S *Astronomy*);
(2) ·0000003595043 (second satellite);
(3) ·0000003447736 (mean value above);
(4) ·0000003301909 (corrected value for distance of first satellite);
(5) ·000000206378 (uncorrected value for distance of first satellite).

3666 & 7729.—(3666.) (By Professor EVANS, M.A.)—The six faces of a cube, each of whose edges is n inches in length, are divided into square inches by two systems of parallel red lines. How many *different* routes of $3n$ inches each, by red lines, are there from one corner of the cube to the corner diagonally opposite?

(7729.) (By B. REYNOLDS, M.A.)—Show that the number of shortest routes from one corner of a chess-board to the opposite one, along the edges of the squares, is 12870.

Solution by the REV. T. C. SIMMONS, M.A.



(3666.) Let ABCD be one of the faces adjacent to A, the starting-point. Then any route which commences in the plane ABCD must cross one of the edges CD, CB.

In either case (let us suppose the former), we have the equivalent of a route traversing a rectangle $2n$ inches long by n inches broad, and whose opposite angles correspond with A and E, the opposite corners of the cube, and we have to consider in how many ways n journeys parallel to one direction can be interspersed among $2n$ journeys parallel to the perpendicular direction. As the extreme lines AB and EF are both available for the former set, this is equivalent to finding in how many ways n indifferent things can be distributed into $2n+1$ different parcels. This is the same as the number of combinations of $3n$ things taken $2n$ at a time (see Prop. 26 of WHITWORTH'S *Choice and Chance*). Hence the total number of routes crossing CD is $3n! / [2n! \, n!]$. Multiplying by 2, we get all the routes traversing the face ABCD, and, multiplying again

by 3, we include all the routes starting from A, which thus amount to $[6 \,.\, 3n!]/[2n!\, n!]$.

It only remains to subtract the routes which have been counted twice. All the routes from D to E, numbering $2n!/n!\,n!$, have been included twice, both among those crossing DC, and among those crossing DF, and similarly for the routes from B to E. The routes from A to C have likewise been counted twice, both among those crossing CD and those crossing CB. There are six of these sets of routes, corresponding to the six corners of the cube lying between A and E. Hence the final result of the number of *different* routes is $6\,(3n!/2n!\,n! - 2n!/n!\,n!)$. It will be seen that each route along the edges has been subtracted twice, as it ought to be, since it is originally included thrice. For $n = 1, 2$, the respective results are 6 and 54, which may be tested by actual calculation.

(7729.) By the same line of reasoning, it will be seen that this question is equivalent to finding in how many ways 8 journeys in one direction can be interspersed among 8 journeys in a perpendicular direction, the extreme ends of the latter being both available for the former. This is the same as the number of combinations of 16 things 8 at a time, *i.e.*, 12870. [For another solution, see Vol. XLII., p. 28.]

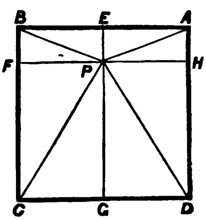
7932 & 7972. (7932.)—(By the EDITOR.)—If $\alpha, \beta, \gamma, \delta$ be the angles subtended by the sides of a square at an internal point not situated in a diagonal, prove that

$$(\tan\alpha + \tan\gamma)^{-1} + (\tan\beta + \tan\delta)^{-1} = (\cot\alpha + \cot\gamma)^{-1} + (\cot\beta + \cot\delta)^{-1} = 1.$$

(7972.) (By Rev. T. C. SIMMONS, M.A. Suggested by Question 7932.) —If the angles of a square ABCD be joined with *any* internal point P, and the angles PAD, PDA, PBC, PCB be denoted respectively by $\alpha, \beta, \gamma, \delta$, prove that

$$(\tan\alpha + \tan\gamma)^{-1} + (\tan\beta + \tan\delta)^{-1} = (\cot\alpha + \cot\beta)^{-1} + (\cot\gamma + \cot\delta)^{-1} = 1.$$

Solution by Rev. E. SKRIMSHIRE, M.A.; Rev. T. GALLIERS, M.A.; *and others.*



(7932.) Draw EPG, FPH parallel to the sides, and take APB $= \alpha$, BPC $= \beta$, CPD $= \gamma$, DPA $= \delta$; then we have

$$\tan\alpha = \frac{(\tan APE + \tan BPE)}{1 - \tan APE \tan BPE} = PE \,.\, AB / (PE^2 - PF \,.\, PH),$$

$$\tan\gamma = PG \,.\, AB / (PG^2 - PF \,.\, PH);$$

whence, putting $PE = e$, $PF = f$, $PG = g$, $PH = h$, $AB = a$,

$$\tan\alpha + \tan\gamma = \frac{ae}{e^2 - fh} + \frac{ag}{g^2 - fh} = \frac{a^2\,(eg - fh)}{(e^2 - fh)(g^2 - fh)}.$$

Adding the reciprocal of this to the reciprocal of a similar expression for $\tan\beta+\tan\delta$, we obtain

$$\frac{1}{\tan\alpha+\tan\gamma}+\frac{1}{\tan\beta+\tan\delta}=\frac{(e^2-fh)(g^2-fh)}{a^2(eg-fh)}+\frac{(f^2-eg)(h^2-eg)}{a^2(fh-eg)}$$

$$=\frac{eg(f^2+h^2)-fh(e^2+g^2)}{a^2(eg-fh)}=\frac{eg(f+h)^2-fh(e+g)^2}{a^2(eg-fh)}=1.$$

Similarly, $\qquad \cot\alpha+\cot\gamma=\dfrac{e^2-fh}{ae}+\dfrac{g^2-fh}{ag}=\dfrac{a(eg-fh)}{aeg}.$

Hence $\qquad \dfrac{1}{\cot\alpha+\cot\gamma}+\dfrac{1}{\cot\beta+\cot\delta}=\dfrac{eg}{eg-fh}+\dfrac{fh}{fh-eg}=1.$

The above proof does not hold when P is situated on either diagonal, since in this case $eg-fh$, which has been used as a factor of numerator and denominator, is equal to 0.

[Put $\angle$ABP $=\theta$, $\angle$ADP $=\phi$; then we have

$$\frac{\text{PB}}{\text{AB}}=\frac{\sin(\alpha+\theta)}{\sin\alpha}=\frac{\text{PB}}{\text{CB}}=\frac{\cos(\beta-\theta)}{\sin\beta},$$

therefore $\qquad \tan\theta=\dfrac{\sin\alpha(\sin\beta-\cos\beta)}{\sin\beta(\sin\alpha-\cos\alpha)}$ ....(1),

similarly $\qquad \tan\phi=\dfrac{\sin\delta(\sin\gamma-\cos\gamma)}{\sin\gamma(\sin\delta-\cos\delta)}$ ....(2).

Again, $\dfrac{\text{AB}}{\text{AP}}=\dfrac{\sin\alpha}{\sin\theta}=\dfrac{\text{AD}}{\text{AP}}=\dfrac{\sin\delta}{\sin\phi}$, and $\dfrac{\text{BC}}{\text{PC}}=\dfrac{\sin\beta}{\cos\theta}=\dfrac{\text{DC}}{\text{PC}}=\dfrac{\sin\gamma}{\cos\phi}$;

therefore $\qquad \dfrac{\tan\theta}{\tan\phi}=\dfrac{\sin\alpha\,.\,\sin\gamma}{\sin\beta\,.\,\sin\delta}$ ....(3).

From the above equations (1), (2), (3), we obtain

$$(\sin\alpha-\cos\alpha)(\sin\gamma-\cos\gamma)=(\sin\beta-\cos\beta)(\sin\delta-\cos\delta),$$

or $\qquad \sin(\alpha+\gamma)-\sin(\beta+\delta)=\cos(\alpha-\gamma)-\cos(\beta-\delta)$ ....(4).

But $\qquad \sin(\alpha+\gamma)=-\sin(\beta+\delta)$, and $\cos(\alpha+\gamma)=\cos(\beta+\delta)$;

whence $\qquad \cos(\alpha-\gamma)-\cos(\beta-\delta)=2[\cos\alpha\cos\gamma-\cos\beta\cos\delta]$

$$=2[\sin\alpha\sin\gamma-\sin\beta\sin\delta].$$

Substituting in (4), this gives

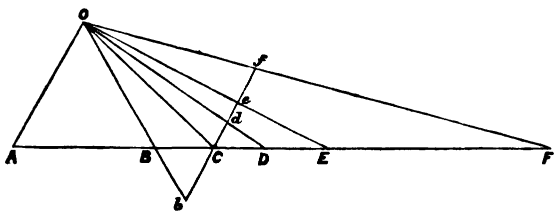
$\sin(\alpha+\gamma)=-\sin(\beta+\delta)=\cos\alpha\cos\gamma-\cos\beta\cos\delta=\sin\alpha\sin\gamma-\sin\beta\sin\delta,$

hence $\dfrac{\cos\alpha\cos\gamma}{\sin(\alpha+\gamma)}+\dfrac{\cos\beta\cos\delta}{\sin(\beta+\delta)}=\dfrac{\sin\alpha\sin\gamma}{\sin(\alpha+\gamma)}+\dfrac{\sin\beta\sin\delta}{\sin(\beta+\delta)}=1$, except in the case when $\sin(\alpha+\gamma)=\sin(\beta+\delta)=0$; that is, when P is on a diagonal.]

(7972.) In this question PAD $=\alpha$, PDA $=\beta$, PBC $=\gamma$, PCB $=\delta$, and AH $\tan\alpha$ + BF $\tan\gamma$ = PH + PF $=a$, whence $\tan\alpha+\tan\gamma=a/e$; so $\tan\beta+\tan\delta=a/g$; hence the stated results follow at once.

7620. (By Rev. T. C. Simmons, M.A.).—(See p. 45 of this Volume.)

Solution by the Proposer.



Take any point O; join OA, OB, OC, and through C draw the transversal $bCdef$ parallel to AO.

Then ABCE, ACDE, ACEF, being harmonic, give respectively $bC = Ce$, $Cd = de$, $Ce = ef$: hence we may take $bC = 2$, $Cd = 1$, $de = 1$, $ef = 2$.

Hence $bd = df$, therefore ABDF is harmonic; and $bCdf$, $bdef$ are obviously harmonic, whence also are BCDF, BDEF.

8046. (By Professor Lloyd Tanner, M.A.)—AP, BP, CP are arcs of great circles bisecting the angles of a spherical triangle ABC; prove

that $$\frac{\sin BPC}{\cos \frac{1}{2}A} = \frac{\sin CPA}{\cos \frac{1}{2}B} = \frac{\sin APB}{\cos \frac{1}{2}C} = \sec r,$$

where r is the radius of the circle inscribed in ABC.

Solutions by (1) Professor Genese, M.A.; (2) W. J. Johnstone, M.A.

1. Draw arc PL perpendicular to BC; then

$$\sin APB : \sin \tfrac{1}{2}A = \sin c : \sin BP, \quad \sin \tfrac{1}{2}A : \sin APC = \sin CP : \sin b,$$

therefore $$\sin APB : \sin APC = \sin c \sin CP : \sin b \sin BP$$
$$= \sin C \sin CP : \sin B \sin BP = \sin C \sin \tfrac{1}{2}B : \sin B \sin \tfrac{1}{2}C$$
$$= \cos \tfrac{1}{2}C : \cos \tfrac{1}{2}B = \sin CPL \cos r : \sin BPL \cos r.$$

But $APB + APC = 2\pi - BPC$; then this angle and PBC are divided into parts whose sines are in the same ratio; therefore $APB = \pi - CPL$ and $APC = \pi - BPL$. So, if PN be perpendicular to AB,

$$BPC = \pi - APN, \quad \text{therefore} \quad \sin BPC = \sin APN = \frac{\cos \frac{1}{2}A}{\cos r};$$

thus $$\frac{\sin BPC}{\cos \frac{1}{2}A} = \sec r = \frac{\sin CPA}{\cos \frac{1}{2}B} = \frac{\sin APB}{\cos \frac{1}{2}C}.$$

Otherwise.—$\sin r = \dfrac{\tan BL}{\tan BPL} = \dfrac{\tan CL}{\tan CPL} = \dfrac{\tan BL + \tan CL}{\tan BPL + \tan CPL}$

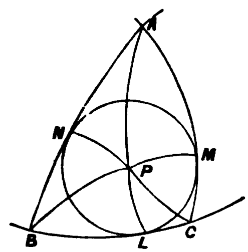
$$= \frac{\sin BC}{\sin BPC} \cdot \frac{\cos BPL \cdot \cos CPL}{\cos BL \cdot \cos CL} = \frac{\sin a}{\sin BPC} \cdot \sin \tfrac{1}{2}B \sin \tfrac{1}{2}C,$$

therefore $\quad \sin r \sin BPC = \sin a \cdot \sin \tfrac{1}{2}B \sin \tfrac{1}{2}C = \tan r \cdot \cos \tfrac{1}{2}A,$

therefore $$\frac{\sin BPC}{\cos \tfrac{1}{2}A} = \sec r.$$

2. $\angle BPL = BPN, \quad CPL = CPM,$

$APN = APM;$



$\therefore \quad BPL + APC = \tfrac{1}{2}$ sum of all these $= \pi$;

therefore $\quad APC = \pi - BPL.$

The right-angled triangle BPL gives

$\cos PBL = \cos PL \sin BPL,$

therefore $\quad \cos \tfrac{1}{2}B = \cos r \sin APC$;

$$\therefore \quad \frac{\sin APC}{\cos \tfrac{1}{2}B} = \sec r = \frac{\sin BPA}{\cos \tfrac{1}{2}C} = \frac{\sin CPB}{\cos \tfrac{1}{2}A}.$$

N.B.—The triangle BPL gives also

$$\tan APC = -\tan BPL = -\frac{\tan BL}{\sin r} = -\frac{\tan (s-b)}{\sin r}.$$

There are similar values for $\tan BPA$ and $\tan CPB$. The sum of these angles $= 2\pi$, so that sum of their tangents = product of their tangents. This gives the relation

$$[\tan (s-a) + \tan (s-b) + \tan (s-c)] \sin^2 r = \tan (s-a) \tan (s-b) \tan (s-c).$$

7946. (By Rev. T. R. TERRY, M.A.)—An inextensible string has one end fixed at the vertex of a cycloid and is wrapped round the outside of the curve, being just long enough to reach as far as a cusp. If the string is unwrapped from the curve and turned round (being continually kept stretched) until it is wrapped round the other half of the cycloid, find the area included between the cycloid and the curve traced out by the moveable end of the string.

Solution by D. EDWARDES ; G. G. STORR, B.A. ; *and others.*

If s be the length of the arc of the cycloid measured from the vertex, and a the radius of the generating circle, an element of area will be $\tfrac{1}{2}(4a-s)^2 d\phi$, or $8a^2(1-\sin \phi)^2 d\phi$, the intrinsic equation of the cycloid being $s = 4a \sin \phi$. Integrating between $\tfrac{1}{2}\pi$ and 0, we have $2a^2(3\pi-8)$.

The area of the semicircle described by the string is $8\pi a^2$; hence the required area $= 8\pi a^2 + 4a^2(3\pi-8) = 4a^2(5\pi-8)$.

7981. (By R. LACHLAN, B.A.)—With any point in the plane of a triangle as centre, three circles can be drawn, so that the angles θ, ϕ, ψ, in which they cut the sides of the triangle, are connected by the relation $\theta \pm \phi \pm \psi = 0$: show that (1) the radius of one of the circles is equal to the sum of the other two; (2) the locus of the centres of such circles having a given radius is a cubic curve whose asymptotes are parallel to the sides of the triangle.

Solution by B. HANUMANTA RAU, M.A.; J. O'REGAN; *and others.*

If α, β, γ be the distances of the point from the sides of the triangle and r the radius of the circle, then

$$\cos\theta = \frac{\alpha}{2r}; \quad \cos\phi = \frac{\beta}{2r}; \quad \cos\psi = \frac{\gamma}{2r}.$$

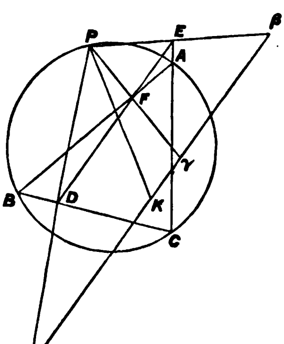
But $$\theta \pm \phi \pm \psi = 0 \quad \text{or} \quad \cos\theta = \cos(\phi \pm \psi);$$

hence $$(\cos\theta - \cos\phi\cos\psi)^2 = (1-\cos^2\phi)(1-\cos^2\psi);$$

therefore $$8r^3 - 2r(\alpha^2+\beta^2+\gamma^2) + \alpha\beta\gamma = 0 \quad \text{......................(1).}$$

If the point is given, α, β, γ are known, and r has three values such that their sum is *zero*; since the three values cannot all be of the same sign, one of them is equal to the sum of the other two. If r is given, (1) represents a *cubic curve* in trilinear coordinates, whose asymptotes are parallel to $\alpha = 0$, $\beta = 0$, $\gamma = 0$, *i.e.*, to the sides of the triangle.

8024. (By R. TUCKER, M.A.)—Prove that the *images* of any point on the circum-circle with respect to the three sides of an inscribed triangle lie on a straight line which passes through the orthocentre.

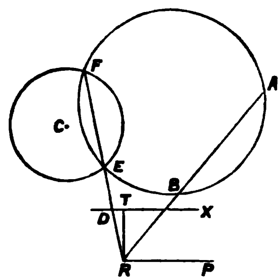


Solution by EMILY PERRIN, B.Sc.; D. BIDDLE; *and others.*

Let ABC be a triangle, K its orthocentre, P a point on its circum-circle; then, if PD, PE, PF be perpendicular to the three sides of the triangle, D, E, F lie on the SIMSON-line of P, which bisects KP. Now, if α, β, γ be the images of P in the three sides, $P\alpha = 2PD$, &c.; hence α, β, γ lie on a straight line through the orthocentre K.

7462. (By the EDITOR.)— Through two given points draw a circle such that its points of intersection with a given circle, and a third given point, shall form the vertices of a triangle of given area.

Solution by Rev. T. C. SIMMONS, M.A.



Through the given points (A, B) draw any circle whose radical axis with the circle C meets AB in R; then R must evidently be a point on the chord of intersection (EF) of C with the required circle. Now we have

ΔPEF = PRF − PRE = a given area;

hence, since PR is given, the difference is also given of the perpendiculars from E and F on PR.

Draw RT perpendicular to PR and equal to this given difference; then, drawing TX parallel to RP, we evidently require to draw from R a line meeting TX in D and the circle in E, F, such that RD = EF.

[This is a much simpler construction for reducing Question 7462 to 7520 than that given in Vol. XL., p. 99.]

7943. (By Rev. T. C. SIMMONS, M.A.)—Prove that the mean value of the n^{th} power of the distance between two points taken at random within a given circle is, according as n is an even positive integer, or an odd integer not less than -1,

$$\frac{2^{n+4}}{n+2}\cdot\frac{1.3.5\ldots(n+1)}{2.4.6\ldots(n+4)}r^n,\qquad \frac{2^{n+5}}{\pi\,(n+2)(n+3)}\cdot\frac{2.4.6\ldots(n+3)}{1.3.5\ldots(n+4)}r^n.$$

Solution by D. EDWARDES; Professor ROY, M.A.; *and others.*

Let O be the centre, P, Q two random points, OP $= x$, PQ $= y$, $\angle$OPQ $= \theta$. While P ranges over the circle, let Q be confined to the concentric circle through P. An element of area at P is $2\pi x dx$, and at Q is $y dy d\theta$. Hence the required average is

$$\frac{2}{\pi^2 r^4}\int_0^r\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\int_0^{2x\cos\theta} y^n \,.\, 2\pi x dx\, y dy\, d\theta$$

(since P may be confined to the concentric circle through Q)

$$=\frac{2^{n+5}}{\pi\,(n+2)\,(n+4)}\,r^n\int_0^{\frac{1}{2}\pi}\cos^{n+2}\theta\, d\theta = \text{the result stated.}$$

3556. (By the EDITOR.)—Show that the equation of the chord common to the conic $ax^2+2hxy+by^2+2gx+2fy=0$ and the circle osculating it at the origin is, θ being the angle between the positive axes,

$$\frac{y}{x}+\frac{2hf+g(a-b)-2af\cos\theta}{2hg+f(b-a)-2bg\cos\theta}=0.$$

Solution by the Rev. T. C. SIMMONS, M.A.

Let the equation of the osculating circle be

$$x^2+2xy\cos\theta+y^2+2mgx+2mfy=0 \quad \text{.....................(1)},$$

and that of the required chord $px+qy=0$ ....(2).

Then for some value of μ we must have

$$(a-\lambda)x^2+2(h-\lambda\cos\theta)xy+(b-\lambda)y^2+2g(1-\lambda m)(gx+fy)=0,$$

identical with $(gx+fy)(px+qy)=0$; hence

$$a-\lambda=gp,\ b-\lambda=fq,\ 1-\lambda m=0,\ 2(h-\lambda\cos\theta)=gq+fp \text{......(3, 4, 5, 6)}.$$

Substitute in (6) the values of p and q obtained from (3) and (4), then

$$2h-2\lambda\cos\theta=\frac{g}{f}(b-\lambda)+\frac{f}{g}(a-\lambda),\ \text{ whence } \lambda=\frac{af^2+bg^2-2fgh}{f^2+g^2-2fg\cos\theta};$$

therefore, in (3) and (4), $\dfrac{p}{q}=\dfrac{f}{g}\cdot\dfrac{a-\lambda}{b-\lambda}=\dfrac{2fh+g(a-b)-2af\cos\theta}{2gh+f(b-a)-2bg\cos\theta}$;

whence, from (2), the required result follows. If the value of m obtained from (5) be substituted in (1), the osculating circle is completely determined.

3247. (By the EDITOR.)—If a set of dominoes be made from double blank up to double n, prove that (1) the number of them whose pips are $n-r$ is the same as the number whose pips are $n+r$; (2) the number is the coefficient of x^{n-r} in the expansion of $(1-x-x^2+x^3)^{-1}$; (3) the total number of dominoes is $\frac{1}{2}(n+1)(n+2)$; (4) if from the dominoes a man is to draw one at random, and to receive as many pounds as there are pips on the domino drawn, the value of his expectation is n pounds.

Solution by the Rev. T. C. SIMMONS, M.A.

1. The different pairs of numbers giving $n+r$ altogether are obtained by putting $p=0, 1, 2,$ &c. in $(n-p)+(r+p)$ until $n-p$ equals either $r+p$ or $r+p+1$. The different pairs giving $n-r$ altogether are obtained by putting $p=0, 1, 2,$ &c. in $(n-r-p)+p$ until $n-r-p$ equals either p or $p+1$. In each case the *number* of different pairs is easily seen to be the greatest integer in $\frac{1}{2}(n-r+2)$.

2. $$(1-x-x^2+x^3)^{-1} = \frac{1}{(1-x)(1-x^2)} = \frac{1+x}{(1-x^2)^2}$$
$$= (1+x)(1+2x^2+3x^4+4x^6+\ldots),$$

in which the coefficient of x^{n-r} is evidently the greatest integer in $\frac{1}{2}(n-r+2)$.

3. The number of ways in which zero and the first n integers can be combined in pairs (excluding doublets) is the same as the number of combinations of $n+1$ things two at a time. Adding the $n+1$ doublets, we obtain for the total number of dominoes $\frac{1}{2}n(n+1)+n+1 = \frac{1}{2}(n+1)(n+2)$.

4. Let p_m denote the number of ways in which m pips can be drawn; then since, from above, $p_{n-r} = p_{n+r}$, we have, for the expectation,

$$\frac{0.p_0+1.p_1+\ldots+(n-1)p_{n-1}+np_n+(n+1)p_{n-1}+\ldots+(2n-1)p_1+2np_0}{p_0+p_1+\ldots+p_{n-1}+p_n+p_{n-1}+\ldots+p_1+p_0}$$
$$= \frac{2n(p_0+p_1+\ldots+p_{n-1})+np_n}{2(p_0+p_1+\ldots+p_{n-1})+p_n} = n.$$

7783. (By Rev. T. C. Simmons, M.A.)—Prove (1) that according as a triangle is obtuse-angled, right-angled, or acute-angled, its nine-point circle will cut, touch, or lie within its circum-circle; (2) having given two circles, of radii R and $\frac{1}{2}$R, not entirely external to each other, an infinite number of triangles can be constructed having the one for circum-circle and the other for nine-point circle respectively.

Solution by B. Hanumanta Rau, B.A.; Professor Matz, M.A.; *and others.*

1. If the triangle is obtuse-angled, the feet of two of the perpendiculars fall without the circum-circle and the foot of the third within. The nine-point circle, therefore, *cuts* the circum-circle. In the right-angled triangle, the vertex coincides with the feet of two perpendiculars and lies on both the circles, and the line joining the centres passes through this common point. The circles therefore touch.

In the case of the acute-angled triangle, all the nine points lie within the circum-circle. Hence the nine-point circle lies entirely within the circum-circle.

2. Let O, O′ be the centres of the two circles, and R, $\frac{1}{2}$R their radii. Divide, at G and P, OO′ internally and externally in the ratio of 2 : 1. Then G is the internal point of similitude of the two circles and the centre of gravity of the required triangle. P, the external point of similitude, is the ortho-centre.

If a be any point on the circle O′, draw Ba, DC at right angles to Oa, cutting the circle O′ again at D and the circle O at BC. BC will be the base of the required triangle, and the point where aG or the perpendicular to BC through D meets the circle O will be the vertex.

Since a is any point on the smaller circle, an infinite number of such triangles can be described. [For other solutions, see Vol. XLIII., p. 37.]

7915. (By Satis Chandra Rây.)—Tangents are drawn to a parabola, so that the intercepts on the tangent at the vertex are in arithmetical progression; prove that the cotangents of the angles of inclination of these tangents to the tangent at the vertex are in harmonic progression.

Solution by W. J. Greenstreet, B.A.; R. Knowles, B.A.; *and others.*

Let the equation to the tangent be $y = mx + am^{-1}$; then this cuts off from $x = 0$ the intercept $\frac{a}{m}$; hence $\frac{a}{m}$, $\frac{a}{m'}$, $\frac{a}{m''}$ are in Arithmetic Progression, and m, m', m'' in Harmonic Progression.

7967. (By Professor Hudson, M.A.)—Find the mass of a ship that would attract an equal ship at a distance of one furlong with a force equal to one pound weight, assuming that the earth is a spherical mass of six thousand trillion tons of four thousand miles radius.

Solution by D. Edwardes; Professor Sarkar, M.A.; *and others.*

Let m denote the mass of the ship, M that of the earth, R its radius. When the ships are at a distance r from each other, the acceleration is $g \cdot \frac{m}{\text{M}} \cdot \frac{\text{R}^2}{r^2}$. If then a ton be the unit of mass, we have

$$\frac{m^2}{\text{M}} \cdot \frac{\text{R}^2}{r^2} \text{ (weight of a ton)} = \text{weight of one pound};$$

where $\text{M} = 6000,000000,000000,000000$, $\text{R} = 4000$, $r = \frac{1}{8}$;

$$\text{therefore} \qquad m^2 = \frac{6000,000000,000000,000000}{64 \times 4000^2 \times 28 \times 4 \times 20},$$

$$m = \frac{5^7}{7} (21)^{\frac{1}{2}} \text{ tons} = 51144 \cdot 75446 \ldots \text{ tons}.$$

[Attraction of earth : attr. of ship $= \frac{\text{M}}{\text{R}^2} : \frac{m}{r^2}$, also $= 2240\,m : 1$].

7888. (By B. HANUMANTA RAU, B.A.)—If A′, B′, C′ be the mid-points of the sides of a triangle ABC, prove that the in-centre of A′B′C′ is collinear with the in-centre and centroid of the triangle ABC.

Solution by R. KNOWLES, B.A.; G. G. STORR, B.A.; *and others.*

The coordinates of the in-centre of A′B′C′ are

$$\Delta\left(\frac{1}{a}-\frac{1}{2s}\right),\ \Delta\left(\frac{1}{b}-\frac{1}{2s}\right),\ \Delta\left(\frac{1}{c}-\frac{1}{2s}\right);$$

those of the in-centre of ABC are each $= \frac{\Delta}{s}$ and the centroid $\frac{2\Delta}{3a}, \frac{2\Delta}{3b}, \frac{2\Delta}{3c}$;

hence the equation to the line joining the in-centres of the two triangles is

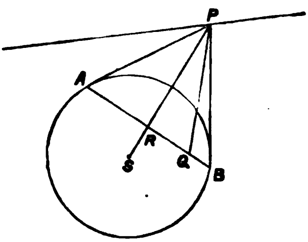
$$\left(\frac{1}{c}-\frac{1}{b}\right)x+\left(\frac{1}{a}-\frac{1}{c}\right)y+\left(\frac{1}{b}-\frac{1}{a}\right)z=0,$$

and this is satisfied by the coordinates of the centroid; therefore the three points are collinear.

[The theorem is evident from the fact that the centroid is the centre of similitude of the two triangles.]

8045. (See p. 107 of this Volume.)

Solution by the Rev. J. J. MILNE, M.A.



First consider the case of a circle, whose centre is S.

Let AB be the polar of P, and let the angle PQR be const. Then the angle SPQ is const.; therefore, since S is a fixed point, and P moves along a fixed straight line, PQ in general envelopes a parabola, focus S, except when the angle PQR is equal to the angle between the given line and the diameter at right angles to it, in which case the angle QPR vanishes, and PQ always passes through the fixed point S. By projection we at once obtain Question 8045.

[This Solution Mr. MILNE sends because he thinks that "the subject of envelopes treated geometrically is the most beautiful part of Geometry, though all reference to it is carefully excluded from text-books."]

APPENDIX.

SOLUTIONS OF SOME OLD QUESTIONS,

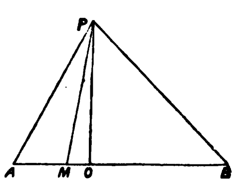
By Âsûtosh Mukhopâdhyây, B.A., F.R.A.S.

1448, 3336, 4171, 6120. (By the late Professor Clifford, F.R.S.) —Find (1) the position of equilibrium of a particle in the plane of a triangle under the resultant attraction (or repulsion) of the perimeter, which is supposed to be formed of matter attracting according to the law of the cube of the distance; and (2) solve the analogous problem for the inverse faces of a tetrahedron.

Solution.

1. We proceed first to find the attraction of a material bar, of uniform thickness and density, on any point, which may be done as follows:—

Let AB be the bar of cross section κ, and density ρ; P the attracted point; PO $(=y)$ the perpendicular from P on AB; $\angle$APO $=\alpha$, $\angle$BPO $=\beta$. Consider the attraction of any element ds at M, on P; let MO $= s$, and $\angle$MPO $=\psi$. The attraction of the element along PM is $\kappa\rho\, ds / \mathrm{PM}^3$.



Now, since $s = y\tan\psi$, we have

$$ds = y\sec^2\psi\, d\psi \text{ and } y = \mathrm{PM}\cos\psi;$$

so that the attraction is $\frac{\kappa\rho}{y^2}\cos\psi\, d\psi$; hence, if X, Y be the total resolved attractions of the bar along AB and PO, we have

$$X = \frac{\kappa\rho}{y^2}\int_{-\beta}^{+\alpha}\sin\psi\cos\psi\, d\psi = \frac{\kappa\rho}{2y^2}(\sin^2\alpha-\sin^2\beta) = \frac{\kappa\rho}{4y^2}(\cos 2\beta-\cos 2\alpha)$$

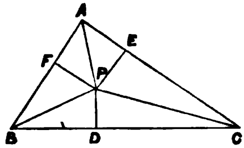
$$Y = \frac{\kappa\rho}{y^2}\int_{-\beta}^{+\alpha}\cos^2\psi\, d\psi = \frac{\kappa\rho}{4y^2}(2\alpha+2\beta+\sin 2\alpha+\sin 2\beta).$$

If l be the length of the bar, it is evident that $y = l/(\tan\alpha-\tan\beta)$;

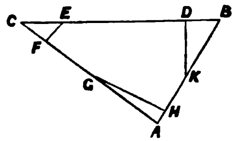
hence $$X = \frac{\kappa\rho}{4l^2}(\tan\alpha-\tan\beta)^2(\cos 2\beta-\cos 2\alpha),$$

$$Y = \frac{\kappa\rho}{4l^2}(\tan\alpha-\tan\beta)^2(2\alpha+2\beta+\sin 2\alpha+\sin 2\beta).$$

Now, consider any triangle ABC, and a point P in its plane; let PD, PE, PF be the perpendiculars on the sides; then, the attraction of the perimeter on P may be represented by two sets of three forces each, viz., one set of three forces, parallel to the sides of the triangle, and having X for their type; the other set at right angles to the sides, and of type Y.



Let the forces be called X_1, X_2, X_3, Y_1, Y_2, Y_3 respectively. Starting from any arbitrary origin D, construct the force-diagram of the system DEFGHK, wherein DE $= X_1$, EF $= Y_2$, FG $= X_2$, GH $= Y_3$, HK $= X_3$, and KD $= Y_1$. The angles at D, F, H are obviously right angles, and the triangle ABC, formed by producing the sides of the polygon, is similar to the given triangle. If we now project the sides of the force-polygon on DE, we have



$$X_1 + (X_2 \cos C - Y_2 \sin C) = X_3 \cos B - Y_3 \sin B \ldots\ldots(1).$$

Thus, projecting any five of the sides on the sixth, in rotation, we get, in all, six equations, of which (1) is the type. Only *four* independent relations, however, can be obtained from this force-diagram, as may be shown by actual calculation, or, better still, from geometrical considerations. Thus, suppose we are given *four* of the elements, viz., X_1, Y_2, X_2, Y_3; then, since all the angles of the polygon are known, we construct DEFGH, and then determine K as the intersection of perpendiculars at H and D. Hence, we have four independent equations (1, 2, 3, 4), involving the six forces, that is, involving the six quantities α, β, γ, δ, ϵ, ζ.

Again, if x, y, z be the trilinear coordinates of P, viz., if PD $= x$, PE $= y$, PF $= z$, we have

$$a = x(\tan\alpha - \tan\beta),\quad b = y(\tan\gamma - \tan\delta),\quad c = z(\tan\epsilon - \tan\zeta)\ldots(5, 6, 7),$$

and we have further the two identical geometrical relations

$$\alpha + \beta + \gamma + \delta + \epsilon + \zeta = 2\pi,\quad ax + by + cz = 2\Delta\ldots\ldots\ldots(8, 9),$$

where Δ is the area of the given triangle. As these nine equations involve the nine unknown quantities α, β, γ, δ, ϵ, ζ, x, y, z, the values of x, y, z can be determined in terms of the known quantities a, b, c, A, B, C, π, Δ. Thus the position of equilibrium of the point is theoretically determined.

2. From the above, it is sufficiently clear that the corresponding case of the tetrahedron is to be solved by the use of tetrahedral coordinates; it is, therefore, enough to indicate how the attraction of any triangular lamina on any point in space may be calculated, according to the law of the inverse cube of the distance. Let AOB be the triangular plate, take O as origin, $\angle$AOB $= \omega$, OA $= a$, OB $= b$, so that the equation of AB is $\frac{x}{a} + \frac{y}{b} = 1$, then the element at any point x, y is $dx\,dy \sin\omega$; hence, if h be the height of the attracted point above the plane, and, α, β be the coordinates of the projection of the point on the plane, we have, for the element of attraction,

$$dA = \mu \sin\omega \frac{dx\,dy}{\left\{h^2 + (x-\alpha)^2 + (y-\beta)^2 + 2(x-\alpha)(y-\beta)\cos\omega\right\}^{\frac{3}{2}}},$$

in which the limits of x are $(a, 0)$, and those of y are $\left\{\frac{b}{a}(a-x),\ 0\right\}$. The total attraction is to be found by resolving this along and perpendicular to the plane.

It may be noticed that the unusual complexity in the first case is solely due to the fact that the law of attraction is that of the inverse cube of the distance. The law of nature gives a very neat and symmetrical answer, even if the densities of the bars be different, viz., if the densities be ρ, σ, τ, the common cross-section κ, and the angles subtended by the bars at the point be 2θ, 2ϕ, 2ψ, we see that the particle is animated by the three forces

$$\frac{2\kappa\rho\sin\theta}{x},\quad \frac{2\kappa\sigma\sin\phi}{y},\quad \frac{2\kappa\tau\sin\psi}{z},$$

the angles between the lines of action being the supplements of θ, ϕ, ψ respectively; hence, for equilibrium, we have $\frac{\rho}{x} = \frac{\sigma}{y} = \frac{\tau}{z}$. If $\rho=\sigma=\tau$, this gives $x = y = z$, or the particle occupies the in-centre, as is also sufficiently obvious from the theorem that the attraction of any bar is the same as that of a certain well-defined circular arc (MINCHIN'S *Statics*, p. 417), which at once shows that the attraction of the perimeter on a particle at the in-centre is the same as the effect of the circumference of the inscribed circle, which effect is, of course, zero.

1507. (By the late Professor CLIFFORD, F.R.S.)—Consider six planes A, B, C, D, E, F, and join the point ABC to the point DEF, and so on; we have thus ten finite straight lines, and their middle points lie in a plane.

Solution.

The easiest way to solve this problem is to regard it as the space-analogue of the well-known proposition in plane geometry, that the middle points of the three diagonals of a complete quadrilateral lie on a right line, which theorem may be re-stated as follows:

"Consider four right lines A, B, C, D, and join the point AB to the point CD, and so on; we have thus three finite straight lines, and their middle points lie in a right line."

It will be noticed that, while in two dimensions we have to deal with four lines, in three dimensions we have $\frac{1}{2}(4 \times 3) = 6$ planes, as it should be.

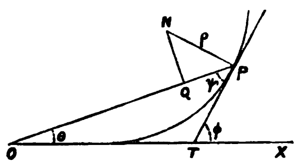
1591. (By Professor HIRST, F.R.S.)—Find the polar equation of a curve whose radii vectores are each divided into segments having a constant ratio, when, upon the same, the respective centres of curvature are projected orthogonally.

Solution.

Let OP be the radius vector, so that $OP = r$, $\angle POX = \theta$. Let $PN = \rho$ = the radius of curvature. Let ψ be the radial angle OPT, and

ϕ the angle PTX which the tangent makes with the prime vector, then

$$PQ = \rho \sin\psi, \quad OQ = r - \rho\sin\psi.$$



Hence, $\quad r - \rho\sin\psi = k\rho\sin\psi,$

where k is the constant ratio. Therefore, we have

$$\frac{r}{1+k} = \rho\sin\psi = \frac{ds}{d\phi}\cdot\frac{rd\theta}{ds}.$$

Hence, $\quad \phi = (1+k)\,\theta,$

the constant of integration vanishing, if θ and ϕ vanish simultaneously, which requires the prime vector to be a tangent to the curve. The curve may be easily traced, if necessary, from the above equation between θ and ϕ. The relation between r and θ is, however, obtained with ease. For $\phi = \theta + \psi$, which gives $k\theta = \psi$; therefore

$$\tan k\theta = \tan\psi = \frac{rd\theta}{dr}, \qquad \text{whence } \frac{dr}{r} = \frac{d\theta}{\tan k\theta}.$$

Integrating, $\qquad \log r = \frac{1}{k}\log\sin k\theta,$

the constant of integration being suppressed, as r and θ vanish together. Hence, the required polar equation is $r^k = \sin k\theta$, and this system, in fact, is, for different values of k, analogous to the family $r^m = a^m\cos m\theta$, when $a = 1$. If $k = 1$, that is, if the radius vector is bisected by the projection of the centre of curvature, the curve is $r = \sin\theta$, a circle, and the property in question gives the theorem of Euc. III. 3. [See Solution of Quest. 1464, Vol. II., p. 65, also p. 19 of Dr. HIRST's *Geometrical Contributions to the "Educational Times."*]

1605. (By the late Professor CLIFFORD, F.R.S.)—Required, the area of the triangle included by three points in space, given by equations of the form $lx + my + nz + sw = 0$.

Solution.

Let a, b, c be the sides of the triangle, and Δ its area; then

$$16\Delta^2 = 2\,(b^2c^2 + c^2a^2 + a^2b^2) - (a^4 + b^4 + c^4).$$

We have now to express the lengths a, b, c in terms of the coefficients in the three given equations which represent the three vertices, in the "four-point coordinate system." But, if

$$\sigma = l + m + n + s, \quad \sigma' = l' + m' + n' + s',$$

we have $$a^2 = \Sigma\left\{\left(\frac{l'}{\sigma'} - \frac{l}{\sigma}\right)\left(\frac{m}{\sigma} - \frac{m'}{\sigma'}\right)AB^2\right\},$$

where AB is an edge of the tetrahedron of reference. (This result is fully worked out in FROST and WOLSTENHOLME's *Solid Geometry*, ed. 1863, pp. 67—69. In the second edition of the work, published in 1875, the result is, however, stated without proof on p. 80.) Hence, by substituting for a^2, b^2, c^2, we see how Δ^2 can be expressed in terms of the constants

involved in the equations to the three points. No very material simplification is effected, even if the three sides coincide with three edges of the tetrahedron of reference. [For a similar problem, see the Solutions of Quest. 1497 in Vol. I., p. 79; Vol. IV., p. 53.]

1691. (By the late Professor CLIFFORD, F.R.S.)—If ρ_1, ρ_2 be the radii of two spheres, and D the distance between their centres, and if a tetrahedron be inscribed in each: prove (1) that the product of the volumes of the tetrahedra into $(D^2-\rho_1^2-\rho_2^2)$ may be expressed as an integral function of the squares of the distances between the vertices of the tetrahedra; and hence (2) deduce the condition $(\Theta = 0)$ that four points in a plane may lie in a circle, and (3), if they do not lie in a circle, state the meaning of Θ.

Solution.

This question relates to and well illustrates the Theory of Powers of Spheres, which is developed in a posthumous memoir, published in CLIFFORD's *Mathematical Papers*, pp. 332—336. The quantity $(D^2-\rho_1^2-\rho_2^2)$ is the squared distance between the centres of the spheres, less the sum of the squares of the radii, which is exactly what has been happily termed the *power* of the two spheres (or, of one of the spheres with regard to the other). Call the given spheres P and A, of radii ρ_1, ρ_2; let BCDE, QRST be the tetrahedra, of volumes V_1, V_2, inscribed in P and A respectively. Then, the vertices of these tetrahedra may be *indiscriminately* regarded, either as so many points, or as so many spheres of infinitesimal radii. Regarding them from the latter point of view, we have to deal with ten spheres, or, rather, with two systems of five spheres, viz., (A, B, C, D, E), (P, Q, R, S, T). Now, writing (AP) to denote the power of the spheres A and P, it is obvious that, since the points B, C, D, E are all on the sphere P, and Q, R, S, T on A, the powers (BP), (CP), (DP), (EP), (AQ), (AR), (AS), (AT) all vanish, so that P and A are the orthogonal spheres of the systems (B, C, D, E) and (Q, R, S, T). But, defining the *apospheric function* of five spheres to be "the product of the tetrahedron, whose vertices are at the centre of any four, into the power (in regard to the fifth) of a sphere cutting them orthogonally," we know that the determinant formed with the powers of two sets of five spheres is equal to 6144 times the product of their apospheric functions. Hence, considering (A, B, C, D, E), (P, Q, R, S, T) as two sets of five spheres, and P, A the orthogonal trajectories of the systems (B, C, D, E), (Q, R, S, T) respectively, and also remembering that the power of A and P = (AP) = $D^2-\rho_1^2-\rho_2^2$, we have

$$(6144)\quad V_1\,(D^2-\rho_1^2-\rho_2^2),\quad V_2\,(D^2-\rho_1^2-\rho_2^2)$$
$$= \begin{vmatrix} (AP) & (AQ) & (AR) & (AS) & (AT) \\ (BP) & (BQ) & (BR) & (BS) & (BT) \\ (CP) & (CQ) & (CR) & (CS) & (CT) \\ (DP) & (DQ) & (DR) & (DS) & (DT) \\ (EP) & (EQ) & (ER) & (ES) & (ET) \end{vmatrix}$$

But it has been shown before that

$$(BP) = (CP) = (DP) = (EP) = (AQ) = (AR) = (AS) = (AT) = 0\,;$$

Hence, dividing both sides by $D^2-\rho_1^2-\rho_2^2 = (AP)$, we have

(6144) $V_1V_2(D^2-\rho_1^2-\rho_2^2) = \mid (BQ), (CR), (DS), (ET) \mid$.

But, since B, C, D, E, Q, R, S, T are all spheres of infinitesimal radii, the powers are nothing but the squares of the lines joining the centres, that is, the squared distances between the points, which themselves are really the vertices of the two tetrahedra. Hence, we see that the product of the volumes of the tetrahedra into $(D^2-\rho_1^2-\rho_2^2)$ is an integral function of the squares of the distances between the vertices of the tetrahedra,—which is the first theorem in question.

In order to deduce the condition that four coplanar points may be concyclic, we notice that, when B, C, D, E are on the same plane, $V_1 = 0$, so that the required condition is

$$\mid (BQ), (CR), (DS), (ET) \mid = 0.$$

This determinant, if developed, would involve the eighth power of the linear magnitude, so that it would appear, at first sight, as if no real geometrical interpretation could be obtained. But it is evident that, without any loss of generality, we may make the two systems of four points coalesce; so that $(BQ) = (CR) = (DS) = (ET) = 0$. Hence, calling the points (B, C, D, E), (1, 2, 3, 4) respectively, the condition that they may be coplanar as well as concyclic becomes

$$\begin{vmatrix} 0 & (12)^2 & (13)^2 & (14)^2 \\ (12)^2 & 0 & (23^2) & (24)^2 \\ (13)^2 & (23)^2 & 0 & (34)^2 \\ (14)^2 & (24)^2 & (34)^2 & 0 \end{vmatrix} = 0;$$

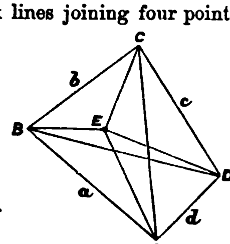
which well-known function is equivalent to

$$\Theta = (12)(34) \pm (13)(42) \pm (14)(23) = 0,$$

the geometrical meaning of which is Ptolemy's Theorem, Euc. VI. D.

When the points do not lie on a circle, the geometric meaning of the relation connecting the mutual distances of any four points in a plane is pointed out in SALMON's *Conics*, ed. 1879, p. 134, Ex. 4. The following, however, is a different interpretation, and includes Ptolemy's Theorem as a particular case.

Let $a, b, c, d, \delta_1, \delta_2$ be the lengths of the six lines joining four points in a plane; let



$$\angle BCD + \angle BAD = 2\phi;$$

make $\angle BCE = \angle ACD$, $\angle CBE = \angle CAD$.

Then, $\angle BEC = \angle ADC$ .... (1),

and, if we join AE, ED, then, from the similar triangles BEC, CDA, we have

$\frac{BC}{BE} = \frac{CA}{AD}$, whence $BE.CA = BC.AD$...(2).

Again, $BC : AC = CE : CD$,

and $\angle BCE + \angle ECA = \angle ACD + \angle ECA$, or $\angle BCA = \angle ECD$,

so that the triangles BCA, ECD are also similar, whence

$\angle CED = \angle CBA$................................ (3);

and $ED : DC = BA : AC$, or $BA.DC = ED.AC$ .... (4).

Again, $\angle A + \angle B + \angle C + \angle D = 4$ right angles;
then, attending to (1) and (3),

$$\angle BED = \angle BCD + \angle BAD = 2\phi.$$

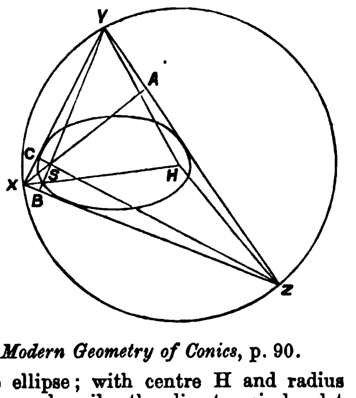
Now, from (2), $BC.AD = BE.CA = \lambda.BE$, say; or, $bd = \lambda.BE$.
From (4), $ac = \lambda.DE$. Also $\delta_1\delta_2 = \lambda.BD$.
Hence, substituting in $BD^2 = BE^2 + ED^2 - 2BE.ED.\cos 2\phi$,
we get $\delta_1^2\,\delta_2^2 = (ac+bd)^2 - 4abcd\cos^2\phi$,
which result is implicitly involved in my Quest. 7754. When $2\phi = \pi$, we get Ptolemy's Theorem. From the above, it appears that, when the points are coplanar without being concyclic, the geometric meaning of the condition Θ may be stated as follows:

"If a, b, c, d, δ_1, δ_2 be the six lines joining four points on a plane, the rectangles ac, bd, $\delta_1\delta_2$ are proportional to the three sides of a triangle, which has an angle equal to the sum of a pair of opposite angles of the quadrilateral formed by the four points, this triangle vanishing when the points are concyclic, so that in this particular case $\delta_1\delta_2 = ac+bd$."

1831. (By Professor Paul Serret.)—Une ellipse et l'un de ses cercles directeurs étant tracés, il existe une infinité de triangles simultanément inscrits au cercle et circonscrits à l'ellipse; le point de rencontre des hauteurs est le même pour tous ces triangles.

Solution.

1. The term Director-circle is ordinarily employed to denote what might perhaps be better called the *ortho-cycle*, viz., the circle-locus of the intersection of tangents at right angles to a conic; this, however, is *not* the sense in which it is used in this question, since, obviously, *no* triangle circumscribing the conic can be inscribed in the circle, which would make the sum of its angles equal to three right angles. But there is another application of the term, the one intended by the Proposer, viz., it denotes *either* of the circles having their centres at the foci, and their radii equal to the transverse axis of the conic. For an able historical review of this double use of the term, which is often perplexing, see Dr. Taylor's admirable work on *The Ancient and Modern Geometry of Conics*, p. 90.



2. Let S and H be the foci of the ellipse; with centre H and radius equal to the major axis of the ellipse, describe the director-circle; let XYZ be a triangle, circumscribing the ellipse and inscribed in the circle. Join HX, HY, HZ; also, join SX, SY, SZ, and produce them to meet the opposite sides in A, B, C respectively. Let $\angle HYS = \beta$, $\angle YBX = \theta$,

and $\angle YBZ = \phi$. Then, from elementary geometry of conics, $\angle HYZ = \angle SYX = \alpha$, say. Since $HY = HZ$, we have $\angle HZY = \angle HYZ = \alpha$, whence $\angle XZS = \angle HZY = \alpha$. Again, $\angle XZS = \angle YXH = \angle XYH = \alpha + \beta$.

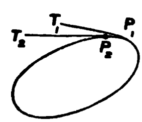
Now, because $\angle CYB = \angle CZB = \alpha$, we see that C, Y, Z, B are concyclic. Therefore, $\angle YCZ = \angle YBZ = \phi$, and $\angle BCZ = \angle BYZ = \alpha + \beta$, which gives $\angle BCS = \alpha + \beta = \angle BXS$, whence B, X, C, S are concyclic, and $\angle YCZ = \angle XBS = \theta$. Hence, we infer that $\theta = \phi$, and, as $\theta + \phi = \pi$, we see that $\theta = \phi$ = one right angle. Thus BY is at right angles to XZ. Similarly, AX and CZ are at right angles to the opposite sides. Hence, S is the orthocentre of the triangle XYZ, and, being a focus of the conic, is a fixed point, which is exactly the theorem in question.

3. If the centre remains fixed, and the ellipse rotates about it, it is obvious that both the foci always lie on the same circle; hence, from the above proof, we have at once the following theorem:—

"If an ellipse rotates about its centre as a fixed point, then the locus of the orthocentres of all triangles circumscribing the ellipse, and inscribed in the double system of its director-circles, is the circle described on the line joining the foci as diameter."

1882. (By the EDITOR.)—Defining the area of a curvilinear figure as by polar coordinates in the Integral Calculus, prove that, if at one end a variable line of constant length touch, in every position, a plane closed re-entering curve of any form consisting of m right and n left loops, the area of the figure traced out by the other end, in the course of a complete revolution, differs from that of the original figure by $(m-n)$ times the area of a circle, whose radius is equal to the constant length of the line.

Solution.



First, suppose that the curve is a single-looped oval, and let P_1, P_2 be two consecutive points on the curve; then P_1T_1, P_2T_2 are two consecutive positions of the touching line, so that $P_1T_1 = P_2T_2 = l$, suppose. If A be the difference of the areas of the loci of P and T, and $d\theta$ the angle between T_1P_1, T_2P_2, we have $2dA = l^2 d\theta$, which gives $2A = l^2 \int d\theta$, the integral being taken round the whole curve. Now, regarding the curve as the limit of a polygon of an infinite number of sides, we see that $d\theta$ is an exterior angle of the polygon, and $\int d\theta$ signifies the sum of all the exterior angles, which sum is known to be 2π, from Euclid I. 32, Cor. 2. Hence, $A = \pi l^2$ = area of a circle whose radius is equal to the constant length of the moving line. If there are more loops than one, viz., m to the right and n to the left, we see that the same argument applies to each of them; whence, attending to the usual convention of signs, we get the difference between the areas of the two loci to be $= (m-n)\pi l^2$.

1927. (By Professor BURNSIDE, M.A.)—Find the conic of least eccentricity which can be drawn through four given points.

Solution.

Let us take for axes of coordinates any two opposite sides of the quadrilateral, which may be produced to meet at O, and include an angle ω; then, if λ_1, λ_2, μ_1, μ_2 be the intercepts which any conic through the four points makes on the axes, we see that, if we make $y=0$ or $x=0$, in its equation, it will reduce to

$$x^2-(\lambda_1+\lambda_2)\,x+\lambda_1\lambda_2=0,\quad y^2-(\mu_1+\mu_2)\,y+\mu_1\mu_2=0\,;$$

whence the equation to the conic is

$$\mu_1\mu_2x^2+2hxy+\lambda_1\lambda_2y^2-\mu_1\mu_2(\lambda_1+\lambda_2)\,x-\lambda_1\lambda_2(\mu_1+\mu_2)\,y+\lambda_1\lambda_2\mu_1\mu_2=0\ldots(1),$$

which may be written in the standard form

$$ax^2+2hxy+by^2+2gx+2fy+c=0,$$

the variable parameter h to be determined from the additional condition that the eccentricity is to be a minimum, e being given, as is well known, by the equation $\dfrac{e^4}{1-e^2}+4=\dfrac{(a+b-2h\cos\omega)^2}{(ab-h^2)\sin\omega}$ ....(2);

whence $\dfrac{de}{dh}=0$ gives the remarkably simple value

$$h=\frac{2ab}{a+b}\cos\omega=\frac{2\lambda_1\lambda_2\mu_1\mu_2}{\lambda_1\lambda_2+\mu_1\mu_2}\cos\omega,$$

which indicates that the solution is unique. By substituting for h in (1), the equation of the required conic is *actually* exhibited. The *actual* value of the eccentricity is also known, for (2) becomes

$$\frac{e^4}{1-e^2}=\frac{(a-b)^2}{ab\sin^2\omega},$$

which is a quadratic in e^2. If the given points are concyclic, $\lambda_1\lambda_2=\mu_1\mu_2=a=b$, whence $e=0$, as it should be. In the particular case, when $\omega=\dfrac{\pi}{2}$, we have $h=0$, and $e^2=\dfrac{a-b}{a}$ or $\dfrac{b-a}{b}$. [The Solver is of opinion that the two solutions of this Question given in Vol. VIII., p. 107—108, "resemble mighty engines set up in vain attempts to kill a poor fly, because, although the Solvers use the methods of Quadric Inversion, Invariants, and Covariants, neither of them actually exhibits the equation to the conic"; whereas this solution "is elementary and straightforward, actually, in a few lines, exhibits the equation of the conic, and gives the value of the eccentricity."]

6661. (By Professor JULLIARD.)—(1) On prend sur la tangente à une courbe fixe, à partir du point de contact, une longueur proportionnelle à la normale en ce point; trouver le lieu de l'extrémité de cette longueur, quand la tangente se déplace. (2) On prend sur la normale à une courbe fixe, à partir du pied de la normale à la courbe, une longueur proportionnelle à la tangente en ce point; trouver le lieu du point ainsi obtenu, quand la normale se déplace. Application aux coniques et à la cycloide.

Solution.

Let the equation to the given curve be $y = f(x)$, referred to any rectangular axes in its plane. Draw the tangent and normal at any point (α, β), and let the lengths of these, as terminated by the axis of x, be T and N respectively. Then, if θ be the angle which the tangent makes with the axis of x, we have

$$\tan\theta = \frac{dy}{dx} = f'(x) = f'(\alpha),$$

where $f'(\alpha)$ means that α is substituted for x in $f'(x)$; also, from geometry,

$$\mathrm{T} = \frac{\beta}{\sin\theta}, \quad \mathrm{N} = \frac{\beta}{\cos\theta}.$$

First Case.—Let us measure on the tangent a length $= k\mathrm{N}$, starting from the point of contact. Then, if X, Y be the coordinates of the point whose locus is sought, we have $\mathrm{X} = \alpha + k\mathrm{N}\cos\theta$, $\mathrm{Y} = \beta + k\mathrm{N}\sin\theta$. Substituting for N and θ their values as given above, we have

$$\mathrm{X} = \alpha + k\beta, \quad \mathrm{Y} = \beta + k\beta f'(\alpha).$$

Since $\beta = f(\alpha)$, this becomes

$$\mathrm{X} = \alpha + kf(\alpha), \quad \mathrm{Y} = f(\alpha)\left\{1 + kf'(\alpha)\right\} \quad \text{..................(A)},$$

from which, if we eliminate α, we get the equation to the required locus.

Second Case.—Let us measure on the normal a length $= k\mathrm{T}$, starting from its foot. Then, if X, Y be the coordinates of the point whose locus is sought, we have $\mathrm{X} = \alpha + k\mathrm{T}\sin\theta$, $\mathrm{Y} = \beta + k\mathrm{T}\cos\theta$. Substituting, as before, for T, θ, β, we get the system,

$$\mathrm{X} = \alpha + kf(\alpha), \quad \mathrm{Y} = f(\alpha)\left\{1 + \frac{k}{f'(\alpha)}\right\} \quad \text{..................(B)},$$

from which, if we eliminate α, we get the equation to the required locus. It will appear in the sequel that, in this case, the presence of $f'(\alpha)$ in the denominator often makes the elimination considerably more complex than in the first case.

Applications.—I. *The Conic.*

(1) Let the conic be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad \text{so that} \quad \frac{dy}{dx} = -\frac{b}{a}\frac{x}{(a^2-x^2)^{\frac{1}{2}}},$$

which makes $\quad f(\alpha) = \dfrac{b}{a}(a^2-\alpha^2)^{\frac{1}{2}}, \quad f'(\alpha) = -\dfrac{b}{a}\dfrac{\alpha}{(a^2-\alpha^2)^{\frac{1}{2}}}.$

Hence, the system (A) becomes

$$x = \alpha + \frac{kb}{a}(a^2-\alpha^2)^{\frac{1}{2}}, \quad y = \frac{b}{a}(a^2-\alpha^2)^{\frac{1}{2}} - \frac{kb^2}{a^2}\alpha,$$

which reduces to

$$\frac{b}{a}(x-ky) = \frac{b}{a}\left(1 + \frac{b^2}{a^2}k^2\right)\alpha, \quad k\,.\,\frac{b^2}{a^2}x + y = \frac{b}{a}\left(1 + \frac{b^2}{a^2}k^2\right)(a^2-\alpha^2)^{\frac{1}{2}};$$

whence, squaring and adding, α is at once eliminated, and the resulting equation, after some algebraical reductions, assumes the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \left(1 + \frac{b^2}{a^2}k^2\right),$$

which shows that the required locus is another ellipse, of equal eccentricity, whose axes are coincident in position with the axes of the old one,

but have been altered in magnitude in the ratio $a : (a^2 + k^2b^2)^{\frac{1}{2}}$. If $k = 1$, that is, if we measure on the tangent a length *equal* to the normal, this ratio becomes $1 : (2-e^2)^{\frac{1}{2}}$.

[In the equilateral hyperbola, we have $a^2 = -b^2$, and, if further $k = 1$, the locus degenerates into the pair of lines $x^2 - y^2 = 0$ which denotes the asymptotes, and this is the well-known theorem that the normal is equal to the intercept on the tangent between the curve and either asymptote. See TAYLOR's *Geometry of Conics*, p. 68.]

(2) For the Second Case, the system (B) becomes

$$x = \alpha + \frac{kb}{a}(a^2-\alpha^2)^{\frac{1}{2}}, \quad y = \frac{b}{a}(a^2-\alpha^2)^{\frac{1}{2}} - k \cdot \frac{a^2-\alpha^2}{\alpha}.$$

If, by transposing and squaring, we get rid of the radicals, we shall have to eliminate α between a quadratic and a biquadratic; this difficulty, however, is easily obviated, as follows. We have, obviously,

$$x - ky = \alpha + k^2 \cdot \frac{a^2-\alpha^2}{\alpha}, \quad \text{or} \quad \alpha^2(1-k^2) - 2\alpha\left(\frac{x-ky}{2}\right) + k^2a^2 = 0.$$

Again, from the first equation,

$$\alpha^2\left(1 + k^2\frac{b^2}{a^2}\right) - 2\alpha \cdot x + (x^2 - k^2b^2) = 0.$$

The result of eliminating α between these two quadratics gives, for the required locus, the quartic curve

$$b^2\left\{4a^2k^2(1-k^2) - (x-ky)^2\right\}\left\{x^2 - (a^2+k^2b^2)\right\}$$
$$= a^2\left\{x(y-kx) + k\left[a^2 + (2k^2-1) \cdot b^2\right]\right\}^2.$$

If $k = 1$, that is, if we measure on the normal a length equal to the tangent, the locus is the quartic $(x^2-b^2)(x-y)^2 - 2a^2x(x-y) + a^2(a^2+b^2) = 0$. Even if we take the particular case of the circle $a^2 = b^2$, no simplification is effected. But, if we take the case of the equilateral hyperbola, where $a^2 = -b^2$, the last term vanishes, and the quartic breaks up into the line $y = x$, and the cubic $(x^2+a^2)(x-y) = 2a^2$. As in the above results we have assumed nothing about the sign of b^2, it is evident that the equations are true for both the ellipse and the hyperbola.

(3) Let us next take the parabola $y^2 = 4ax$.

Here $\quad \dfrac{dy}{dx} = \left(\dfrac{a}{x}\right)^{\frac{1}{2}}; \quad \therefore f(\alpha) = 2(a\alpha)^{\frac{1}{2}}, \; f'(\alpha) = \left(\dfrac{a}{\alpha}\right)^{\frac{1}{2}}.$

Hence, the system (A) becomes $x = \alpha + 2k(a\alpha)^{\frac{1}{2}}$, $y = 2(a\alpha)^{\frac{1}{2}} + 2ka$.

The result of eliminating α easily gives, for the locus $y^2 = 4a(x + ak)$, another parabola, of equal latus rectum, and whose vertex is at a distance ak from that of the first.

(4) For the Second Case, the system (B) becomes

$$x = \alpha + 2k(a\alpha)^{\frac{1}{2}}, \; y = 2(a\alpha)^{\frac{1}{2}} + 2ka,$$

whence $\quad x - ky = \alpha(1-2k^2), \quad 2kx - y = 2(a\alpha)^{\frac{1}{2}}(2ky-1),$

which, by the elimination of α, leads to $(2kx-y)^2 = 4a(1-2k^2)(x-ky)$, another parabola, passing through the origin, the latus rectum of which is $6ak(2k^2-1)/(4k^2+1)$, the axis is the line $2kx - y = 6ak(1-2k^2)/(1+4k^2)$, the directrix is the line $x + 2ky + a(1+k^2) = 0$, the coordinates of the focus are $a(1-5k^2)/(1+4k^2)$, and $2ak(k^2-2)/(1+4k^2)$, so that the locus of the focus of the parabolic locus is a right line for different values of a, and the cubic $\eta^2(4\xi + 5a) = 4(a-\xi)(a+\xi)^2$, for different values of k. In the

particular case, when $k = 2^{-\frac{1}{2}}$, the locus degenerates into the line $y = 2^{\frac{1}{2}} . x$.

II. *The Cycloid.*

The differential equation of the cycloid, referred to its vertex as origin, is $\left(\frac{dy}{dx}\right)^2 = \frac{2a-x}{x}$, and the integral is

$$y = a \text{ vers}^{-1} \frac{x}{a} + (2ax - x^2)^{\frac{1}{2}},$$

so that $\quad f(\alpha) = a \text{ vers}^{-1} \frac{x}{\alpha} + (2a\alpha - \alpha^2)^{\frac{1}{2}}, \; f'(\alpha) = \left(\frac{2a-\alpha}{\alpha}\right)^{\frac{1}{2}}.$

If we now substitute in Systems (A) and (B), and eliminate α, we shall get for the loci two transcendental curves.

6682. (By Professor Eddy, M.A.)—If E^2 be the sum of the squares of the edges of a tetrahedron, F^2 the sum of the squares of the areas of the faces, and V the volume; prove that the principal semi-axes of the ellipsoid inscribed in the tetrahedron, touching each face at its centroid, and having its centre at the centroid of the tetrahedron, are the roots of

$$k^6 - \frac{E^2}{2^4 . 3} k^4 + \frac{F^2}{2^4 . 3^2} k^2 - \frac{V^2}{2^6 . 3} = 0.$$

Solution.

Take the centroid of the tetrahedron as the origin; then, according to the notation of Tait's Quaternions, § 252, the ellipsoid is $S\rho\,\phi\rho = 1$. Again, if $\alpha, \beta, \gamma, \delta$ be the vectors from the origin to the vertices of the tetrahedron, we have $\alpha + \beta + \gamma + \delta = 0$. Now, from elementary mechanical principles, the vector from the origin to the centroid of the face opposite the vector α is known to be $-\frac{1}{3}\alpha$; so that the perpendicular from the origin on this face is easily found to be

$$[\phi(-\tfrac{1}{3}\alpha)]^{-1} = [V(\beta\gamma + \gamma\delta + \delta\beta)]^{-1} S\beta\gamma\delta = [V(\alpha\beta - 3\beta\gamma + \gamma\alpha)]^{-1} S\alpha\beta\gamma.$$

Therefore $\qquad \phi\alpha = 3V(3\beta\gamma - \gamma\alpha - \alpha\beta) S^{-1}\alpha\beta\gamma.$

The corresponding symmetrical equations for β, γ are at once written down, viz., we have

$$\phi\beta = 3V(3\gamma\alpha - \alpha\beta - \beta\gamma) S^{-1}\alpha\beta\gamma, \quad \phi\gamma = 3V(3\alpha\beta - \beta\gamma - \gamma\alpha) S^{-1}\alpha\beta\gamma.$$

For the axes, $\phi\rho$ must be codirectional with ρ, which condition leads to $(\phi + k^{-2})\rho = 0$. Now, selecting $S\rho$ as the operator, we have to determine k from the equation $k^2 = T^2\rho$. Hence, the discriminating cubic is

$$S(\phi + k^{-2})\alpha\,(\phi + k^{-2})\beta\,(\phi + k^{-2})\gamma = 0,$$

which is obviously equivalent to

$$k^{-6} + Pk^{-4} + Qk^{-2} + R = 0, \quad \text{or } k^6 + \frac{Q}{R} k^4 + \frac{P}{R} k^2 + \frac{1}{R} = 0,$$

where the values of P, Q, R are given by

$$P = \frac{S(\alpha\beta\phi\gamma + \beta\gamma\phi\alpha + \gamma\alpha\phi\beta)}{S\alpha\beta\gamma},$$

$$Q = \frac{S(\alpha\phi\beta\phi\gamma + \beta\phi\gamma\phi\alpha + \gamma\phi\alpha\phi\beta)}{S\alpha\beta\gamma}, \quad R = \frac{S\phi\alpha\phi\beta\phi\gamma}{S\alpha\beta\gamma}.$$

$$\text{Now, } P = \frac{3}{S^2\alpha\beta\gamma}\left\{\begin{array}{r} 3\,(V^2\alpha\beta + V^2\beta\gamma + V^2\gamma\alpha) \\ -2\,(S\beta\gamma\, V\alpha\beta + S\gamma\alpha\, V\beta\gamma + S\alpha\beta\, V\gamma\alpha)\end{array}\right\}$$

$$= \frac{3}{4S^2\alpha\beta\gamma}\left\{\begin{array}{l} V^2(\beta-\gamma)(\gamma-\alpha) + V^2(\beta-\alpha)(\delta-\alpha) \\ +V^2(\delta-\alpha)(\gamma-\alpha) + V^2(\beta-\delta)(\gamma-\delta)\end{array}\right\} = -\frac{4F^2}{3V^2};$$

similarly $$Q = -\frac{72}{S^2\alpha\beta\gamma} S(\alpha^2+\beta^2+\gamma^2+\alpha\beta+\beta\gamma+\gamma\alpha)$$

$$= -\frac{9}{S^2\alpha\beta\gamma}\left\{\begin{array}{c}(\alpha-\beta)^2+(\beta-\gamma)^2+(\gamma-\alpha)^2+(\alpha-\delta)^2 \\ +(\beta-\delta)^2+(\gamma-\delta)^2\end{array}\right\} = \frac{2^2E^2}{V^2},$$

$$R = \frac{-2^4\,.\,3^3}{S^2\alpha\beta\gamma} = -\frac{2^6\,.\,3}{V^2}.$$

Therefore $$\frac{Q}{R} = -\frac{E^2}{2^4\,.\,3},\quad \frac{P}{R} = \frac{F^2}{2^4\,.\,3^2},\quad \frac{1}{R} = -\frac{V^2}{2^6\,.\,3}.$$

Hence, finally, we get

$$k^6 - \frac{E^2}{2^4\,.\,3},\ k^4 + \frac{F^2}{2^4\,.\,3^2},\ k^2 - \frac{V^2}{2^6\,.\,3} = 0.$$

6664. (By Professor MATZ, M.A.)—Find the centroid, (1) of the arc of a leaf, (2) of the surface of a leaf, of the curve whose polar equation is $\rho = m^2(1-\sin 2\theta)(1+\sin 2\theta)^{-1}$.

Solution.

The curve is $$r^2 = m^2 . \frac{1-\sin 2\theta}{1+\sin 2\theta} \quad\text{.................................(1).}$$

If $\bar{x}$, $\bar{y}$ be the coordinates of the centroid of the arc of the leaf, we have $$\bar{x} = \frac{\int x\,ds}{\int ds},\quad \bar{y} = \frac{\int y\,ds}{\int ds}.$$

Now, assume $\theta = \frac{1}{4}\pi - \psi$, so that $\sin 2\theta = \cos 2\psi$, and $d\theta = -d\psi$, which leads to $r^2 = m^2\tan^2\psi$.

Therefore $$\frac{dr}{d\theta} = -\frac{m}{\cos^2\psi} = \frac{-2m}{(\cos\theta+\sin\theta)^2} = \frac{-2m}{(1+\sin 2\theta)};$$

therefore $$\left(\frac{ds}{d\theta}\right)^2 = r^2 + \left(\frac{dr}{d\theta}\right)^2 = m^2 . \frac{5-\sin^2 2\theta}{(1+\sin 2\theta)^2},$$

therefore $$\int ds = m\int\frac{(5-\sin^2 2\theta)^{\frac{1}{2}}}{1+\sin 2\theta}\,d\theta.$$

Putting $\sin 2\theta = 5^{\frac{1}{2}}\sin\phi$, $2\cos 2\theta\, d\theta = 5^{\frac{1}{2}}\cos\phi\, d\phi$, $\cos 2\theta = (1-5\sin^2\phi)^{\frac{1}{2}}$,

this is easily transformed into $$s = \frac{5m}{2}\int\frac{\cos^2\phi\, d\phi}{(1+5^{\frac{1}{2}}\sin\phi)(1-5\sin^2\phi)^{\frac{1}{2}}},$$

which is a complicated hyperelliptic form, and cannot be integrated in a finite form. Similarly, since $x = r\cos\theta$, we have

$$\int x\,ds = m^2\int\frac{(1-\sin 2\theta)^{\frac{1}{2}}(5-\sin^2 2\theta)^{\frac{1}{2}}}{(1+\sin 2\theta)^{\frac{3}{2}}}\cos\theta\, d\theta,$$

which, by assuming $\sin 2\theta = 5^{\frac{1}{2}} \sin\phi$, may, as before, be put into the form

$$\int x\,ds = \frac{5m^2}{2.2^{\frac{1}{2}}} \int \left(\frac{\cos\phi}{1+5^{\frac{1}{2}}\sin\phi}\right)^2 \left[1+(1-5\sin^2\phi)^{\frac{1}{2}}\right]^{\frac{1}{2}} d\phi,$$

which is at least as complicated as the preceding one. It appears, therefore, that the values of the coordinates of the centroid of the arc of the leaf cannot be expressed in a finite form.

In order to find the centroid of the area of the leaf, we have

$$\bar{x} = \tfrac{2}{3}\frac{\int r^3\cos\theta\,d\theta}{\int r^2\,d\theta}, \quad \bar{y} = \tfrac{2}{3}\frac{\int r^3\sin\theta\,d\theta}{\int r^2\,d\theta}.$$

Putting, as before, $\theta = \frac{1}{4}\pi - \psi$, and $r = m\tan\psi$, these are transformed into

$$\bar{x} = \frac{2^{\frac{1}{2}}}{3} m \frac{\int \tan^3\psi\,(\cos\psi+\sin\psi)\,d\psi}{\int \tan^2\psi\,d\psi},$$

$$\bar{y} = \frac{2^{\frac{1}{2}}}{3} m \frac{\int \tan^3\psi\,(\cos\psi-\sin\psi)\,d\psi}{\int \tan^2\psi\,d\psi}.$$

Since $\int \tan^3\psi\,(\cos\psi+\sin\psi)\,d\psi = \int \frac{\sin^3\psi}{\cos^2\psi}\,d\psi + \int \frac{\sin^4\psi}{\cos^3\psi}\,d\psi$

$$= \cos\psi + \sec\psi - \frac{\sin^3\psi}{\cos^2\psi} + \tfrac{3}{2}\frac{\sin\psi}{\cos^2\psi} - \tfrac{3}{2}\log\tan\left(\tfrac{1}{4}\pi + \tfrac{1}{2}\psi\right)$$

$$\int \tan^2\psi\,d\psi = \tan\psi - \psi,$$

$$\int \tan^3\psi\,(\cos\psi-\sin\psi)\,d\psi$$

$$= \cos\psi + \sec\psi + \frac{\sin^3\psi}{\cos^2\psi} - \tfrac{3}{2}\frac{\sin\psi}{\cos^2\psi} + \tfrac{3}{2}\log\tan\left(\tfrac{1}{4}\pi + \tfrac{1}{2}\psi\right),$$

we can always find $\bar{x}$, $\bar{y}$, in a finite form, for any assigned limits of θ, ψ. For example, if the limits of θ are $(\frac{1}{2}\pi, \frac{1}{4}\pi)$, those of ψ $(-\frac{1}{4}\pi, 0)$, and the corresponding values of $\bar{x}$, $\bar{y}$ are

$$\bar{x} = \frac{2^{\frac{1}{2}}}{3}\cdot\frac{8+6\log(\sqrt{2}-1)}{4-\pi}m, \quad \bar{y} = \frac{2^{\frac{1}{2}}}{3}\cdot\frac{8(1-\sqrt{2})+6\log(\sqrt{2}+1)}{4-\pi}m.$$

6788. (By C. B. S. Cavallin, M.A.)—Find the position in space for a triangle of given dimensions, in order that the sum of the times required for particles to descend down its sides may be a minimum.

Solution.

Let the plane of the triangle be inclined at an angle ϕ to the vertical plane; assume $\frac{1}{2}(\pi - A) + \theta$ to be the inclination of AB to the line of greatest slope through A on the plane. The resolved part of gravity along this line of greatest slope is $g\cos\phi$; and the whole time of falling

down the three sides of the triangle is given by

$$T = \left(\frac{2}{g\cos\phi}\right)^{\frac{1}{2}} \left\{ \left(\frac{a}{\sin(\frac{1}{2}A-\theta)}\right)^{\frac{1}{2}} + \left(\frac{b}{\sin(\frac{1}{2}A+\theta)}\right)^{\frac{1}{2}} + \left(\frac{c}{\sin(\frac{1}{2}A+C+\theta)}\right)^{\frac{1}{2}} \right\}.$$

In order to determine when T is a maximum or a minimum, we have $\frac{dT}{d\phi} = 0$, which gives $\phi = 0$, showing that the plane of the triangle must be vertical. Moreover, θ is determined from the equation $\frac{dT}{d\theta} = 0$, which gives

$$\cot(\tfrac{1}{2}A-\theta)\left[a\operatorname{cosec}(\tfrac{1}{2}A-\theta)\right]^{\frac{1}{2}}$$
$$= \cot(\tfrac{1}{2}A+\theta)\left[b\operatorname{cosec}(\tfrac{1}{2}A+\theta)\right]^{\frac{1}{2}} + \cot(C+\tfrac{1}{2}A+\theta)\left[c\operatorname{cosec}(C+\tfrac{1}{2}A+\theta)\right]^{\frac{1}{2}},$$

whence θ may be determined.

6885. (By H. Fortey, M.A.)—Find the number of different rows that can be made with r_1 indifferent balls of one colour, r_2 of another colour, r_3 of a third colour, &c. (all the balls being used in each row), in which no two balls of the same colour are in contact.

Solution.

Let us begin with the simple case where there are n balls, of which two are white, three black, and all the rest of other different colours. Suppose for a moment that balls of the same colour are distinguishable, and call the white balls w_1, w_2. Now the whole number of rows is $n!$ and w_1, w_2 will come together in that order in J $(n!)$ rows, and in the order w_2, w_1 also in J $(n!)$ rows. Therefore the number of rows in which the white balls are separated is $(1-2J)\,n!$. Similarly, had we begun by separating the black balls, we should have found the number of rows to be

$$(1-6J+6J^2)\,n!\,;$$

and it is obvious (or, if not, can be proved) that the number of rows in which both white *and* black balls are separated is

$$(1-2J)(1-6J+6J^2)\,n!\,,$$

where it is immaterial in what order the operating functions are written. If balls of the same colour are indifferent, the result is

$$\frac{(1-2J)(1-6J+6J^2)\,n!}{2!\;3!}.$$

Let $\phi_2, \phi_3, \ldots \phi_r$ be the operative symbols for separating 2, 3, ... r balls, then $\phi_2 = 1-2J$, $\phi_3 = 1-6J+6J^2$, and when ϕ_r has been determined the problem is solved.

Now, suppose there are $n+r$ balls, of which $(r+1)$ are black and $(n-1)$ are white. Then, if balls of the same colour are indifferent, the number of rows with the black balls all separated is $\frac{\phi_{r+1}\,\overline{n+r}\,!}{\overline{n-1}\,!\;\overline{r+1}\,!}$. But, looking at the question from another point of view, we see that in the line

$$.\,a\,.\,a\,.\,a\,.\,a\,.\,a\,.\,a\,.\,a\,.$$

if the a's represent the $(n-1)$ white balls, the black balls may be placed one each on any of the n dots. Therefore

$$\frac{\phi_{r+1}\,\overline{n+r}\,!}{\overline{n-1}\,!\;\overline{r+1}} \equiv {}^nC_{r+1} \equiv \frac{n(n-1)\ldots(n-r)}{\overline{r+1}\,!},$$

therefore $\qquad \phi_{r+1}\, \underline{n+r}! \equiv n(n-1)\dots(n-r)\, \underline{n-1}!.$

Assume that $\qquad \phi_{r+1} \equiv a_1 + a_2 J + a_3 J^2 + \dots + a_{r+1} J^r,$

then $\qquad (a_1 + a_2 J + \dots + a_{r+1} J^r)\, \underline{n+r}! \equiv n(n-1)\dots(n-r)\, \underline{n-1}!,$

$$\therefore\; n(n-1)\dots(n-r) \equiv a_1 (n+r)(n+r-1)\dots n + a_2 (n+r-1)\dots n + \&c. + a_r (n+1) n + a_{r+1} n.$$

Or, in factorial notation,

$$n^{(r+1)} \equiv a_1 (n+r)^{(r+1)} + a_2 (n+r-1)^{(r)} + \dots$$
$$\dots + a_{r-1} (n+2)^{(3)} + a_r (n+1)^{(2)} + a_{r+1} n.$$

Now operate on both sides with Δ^{r-s+2}, and we have

$$(r+1) r \dots s \,.\, n^{(s-1)} \equiv (r+1) r \dots s\, (n+r)^{(s-1)} a_1$$
$$+ r(r-1)\dots(s-1)(n+r-1)^{(s-2)} a_2 + \&c.$$
$$+ (r-s+3)(r-s+2)\dots 2\,(n+r-s+2)\, a_{s-1}$$
$$+ (r-s+2)!\; a_s.$$

In this identity, make $n = -r+s-2$; then all the terms on the right-hand side vanish, except the last, and we have

$$(r-s+2)!\; a_s = (r+1) r \dots s\, (-r+s-2)^{(s-1)},$$

and, by reduction, $\quad a_s = (-)^{s-1} \dfrac{r!\; \underline{r+1}!}{\underline{r-s+2}!\; \underline{r-s+1}!\; \underline{s-1}!},$

$$\therefore\quad a_1 = 1, \qquad a_4 = -\frac{[r(r-1)]^2 (r+1)(r-2)}{3!},$$
$$a_2 = -r(r+1),$$
$$a_3 = \frac{r^2 (r+1)(r-1)}{2!}, \qquad a_5 = \frac{[r(r-1)(r-2)]^2 (r+1)(r-3)}{4!},$$
$$\&c. = \&c.;$$

therefore

$$\phi_{r+1} = 1 - r(r+1) J + \frac{r^2 (r+1)(r-1)}{2!} J^2 - \frac{[r(r-1)]^2 (r+1)(r-2)}{3!} J^3 + \&c.,$$

$$\text{or } \phi_r = 1 - (r-1) r J + \frac{(r-1)^2 r (r-2)}{2!} J^2 - \frac{[(r-1)(r-2)]^2 r (r+3)}{3!} J^3 + \&c.$$

If therefore there are m sets of balls, r_1 of one colour, r_2 of another, &c., and $r_1 + r_2 + \dots + r_m = n$, then, balls of the same colour being indifferent, the number of rows in which no two balls of the same colour are in contact is

$$\frac{\phi_{r_1} \phi_{r_2} \phi_{r_3} \dots \phi_{r_m} \,.\, n!}{r_1!\; r_2!\; r_3! \dots r_m!}.$$

As an application of the general formula, suppose there are 10 balls, of which 4 are white, 3 black, 2 red, and 1 blue, then the number of rows will be

$$\frac{\phi_2 \phi_3 \phi_4 \,.\, 10!}{2!\; 3!\; 4!}.$$

Now $\qquad \phi_4 = 1 - 12J + 36J^2 - 24J^3;$

therefore $\qquad \phi_4 \,.\, 10! = 10! - 12 \,.\, 9! + 36 \,.\, 8! - 24 \,.\, 7!,$

$$\therefore\quad \phi_3 \phi_4 \,.\, 10! = 10! - 6 \,.\, 9! + 6 \,.\, 8! - 12\,(9! - 6 \,.\, 8! + 6 \,.\, 7!)$$
$$+ 36\,(8! - 6 \,.\, 7! + 6 \,.\, 6!) - 24\,(7! - 6 \,.\, 6! + 6 \,.\, 5!)$$
$$= 10! - 18 \,.\, 9! + 114 \,.\, 8! - 312 \,.\, 7! + 360 \,.\, 6! - 144 \,.\, 5!;$$

$$\therefore\ \phi_2\phi_3\phi_4 \,.\, 10! = 10!-2\,.\,9!-18\,(9!-2\,.\,8!)+114\,(8!-2\,.\,7!)$$
$$-312\,(7!-2\,.\,6!)+360\,(6!-2\,.\,5!)-144\,(5!-2\,.\,4!)$$
$$= 10!-20\,.\,9!+150\,.\,8!-540\,.\,7!+984\,.\,6!-864\,.\,5!+288\,.\,4!$$
$$= 12888\,.\,4!;$$
$$\therefore\ \frac{\phi_2\phi_3\phi_4\,.\,10!}{2!\,3!\,4!} = \frac{12888\,.\,4!}{2!\,3!\,4!} = 1074,$$

which is the number of rows in which no two balls of the same colour are in contact.

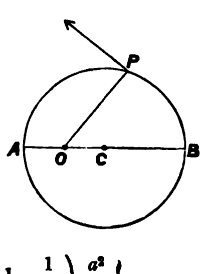
7132. (By N. Nicolls, B.A.)—A van of height b open in front is moved forward with a given uniform velocity V; if the rain descending vertically strike the floor of the van at a distance a from the front, find the velocity of the rain as it strikes the floor.

Solution.

Let v be the velocity of the rain. Impress on the rain-drop as well as on the carriage a velocity V, equal and opposite to that of the carriage. Then, by the parallelogram law, we have $\dfrac{v}{V} = \dfrac{b}{a}$.

7337. (By H. L. Orchard, M.A.)—P is a particle moving with uniform angular velocity, ω, in the circumference of a circle of radius a and centre C. If O be any point in the plane of the circle such that $CO = a \sin 45°$, find the maximum angular velocity of P with regard to O.

Solution.



Let $CP = a$, $CO = \dfrac{a}{n}$, $OP = r$, $v =$ P's linear velocity, so that $v = \omega a$. Then the component linear velocity of P, at right angles to OP, is $v \cos OPC$; then the angular velocity about O is $$\Omega = \frac{v}{r}\cos OPC.$$

Now, $$\cos OPC = \frac{r^2 + \left(1-\dfrac{1}{n^2}\right)a^2}{2ar};$$

therefore $$\Omega = \frac{v}{2a}\left\{1 + \left(1-\frac{1}{n^2}\right)\frac{a^2}{r^2}\right\} = \frac{\omega}{2}\left\{1+\left(1-\frac{1}{n^2}\right)\frac{a^2}{r^2}\right\}.$$

Hence Ω will be a maximum, when r is a minimum and equal to
$$AO = a\left(1-\frac{1}{n}\right) = a\,\frac{n-1}{n},$$

which gives $\Omega = \omega \dfrac{n}{n-1}$. Similarly, the minimum value of Ω is found to be $= \omega \dfrac{n}{n+1}$. In the particular case, when $CO = a \sin 45°$, we have $n = 2^{\frac{1}{2}}$, and the maximum and minimum values of Ω are

$$\omega(2+2^{\frac{1}{2}}) \text{ and } \omega(2-2^{\frac{1}{2}}),$$

respectively. If Ω_1, Ω_2 be the maximum and minimum values of Ω, we have always the relation $$\frac{1}{\Omega_1} + \frac{1}{\Omega_2} = \frac{2}{\omega},$$

showing that Ω_1, ω, Ω_2 are in harmonic progression.

7436. (By Asûtosh Mukhopâdhyây.)—Is the expression $i^{h^{m/n}}$, where $i^2 = -1$, real for any values of h, m, n? If so, discriminate the cases.

Solution.

It is obvious that $i^{h^{m/n}}$ is real, whenever $h^{m/n}$ is an even integer, positive or negative, of the form $\pm 2p$; and there is an infinite number of ways in which this condition can be satisfied. Again, $\cos\theta + i\sin\theta = e^{i\theta}$. Putting $\theta = \frac{1}{2}\pi$, we get $i = e^{\frac{1}{2}\pi i}$, therefore $i^{h^{m/n}} = e^{\frac{1}{2}\pi i h^{m/n}}$. Hence, the expression under consideration is real, whenever $h^{m/n} = ki$, where k is any *real* number, positive or negative, integral or fractional. It is easy to see that this latter condition follows at once from the general theorem that

$$(a+bi)^{p+qi} = r^p e^{-q\theta}\left[\cos(p\theta + q\log r) + i\sin(p\theta + q\log r)\right],$$

where $$r^2 = a^2 + b^2, \quad \theta = \tan^{-1}\frac{b}{a}.$$

7894. (By Professor Hudson, M.A.)—Prove that, in the steady motion in one plane of a uniform incompressible fluid under the action of natural forces, if u, v be the velocities at x, y, parallel to the axes,

$$v\left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2}\right)u - u\left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2}\right)v = 0.$$

Solution.

The "equation of continuity" for the uniplanar motion of an incompressible fluid is $$\frac{du}{dx} + \frac{dv}{dy} = 0 \quad \text{.................................(1).}$$

If the motion is also steady, $\dfrac{du}{dt} = \dfrac{dv}{dt} = 0$,

whence the equations of motion are

$$u\frac{du}{dx}+v\frac{du}{dy}=\text{X}-\frac{1}{\rho}\frac{dp}{dx},\qquad u\frac{dv}{dx}+v\frac{dv}{dy}=\text{Y}-\frac{1}{\rho}\frac{dp}{dy}.$$

If, in addition, the forces are such as occur in nature, they have a potential V, viz., $\text{X}=-\frac{d\text{V}}{dx}$, $\text{Y}=-\frac{d\text{V}}{dy}$. Then, writing $dp=\rho\, d\text{P}$, and assuming $\text{Q}=\text{P}+\text{V}$, the equations of motion reduce to

$$u\frac{du}{dx}+v\frac{du}{dy}=-\frac{d\text{Q}}{dx},\qquad u\frac{dv}{dx}+v\frac{dv}{dy}=-\frac{d\text{Q}}{dy}\ \ldots\ldots\ldots\ldots(2,\ 3),$$

By substituting $-\frac{dv}{dy}$ for $\frac{du}{dx}$, and differentiating (2) and (3) with regard to y and x respectively, Q is at once eliminated, and we get

$$v\left(\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}\right)=u\left(\frac{d^2v}{dx^2}+\frac{d^2v}{dy^2}\right),$$

which is the relation required.

[If ∇^2 denote LAPLACE's Operator, this may be written $v\nabla^2u=u\nabla^2v$.]

NOTE ON QUESTION 6960.

As to Dr. MACALISTER's remark on my solution of his Quest. 6960 (Vol. XLII., p. 110), the phrase "time variation of the *position* of the momentum," to which he takes objection, is an exact paraphrase of his own words, "rate per second at which momentum is *deflected*." I used the word *position* to signify *angular position* or *direction*, and I took the *deflection of momentum* to mean the "change of direction of the momentum," and what the theorem asserts is that, the normal force is measured by the time-variation of the direction of the momentum; that is, by the rate per unit of time at which the direction of momentum is changing, or at which momentum is being deflected. Of course, the *position* (= angular position = direction) of the momentum does not necessarily increase the *position* (= distance from origin whence s is measured) of the particle.

8049. (By Professor HUDSON, M.A.)—Find the locus of the vertex of a parabola of which the axis is parallel to that of a given catenary with which it has contact of the second order.

Solution.

Refer the system to two rectangular axes, the y-axis being vertically upwards through the lowest point of the catenary, the axis of x horizontal, and the origin at a distance c below the lowest point, where c is the parameter of the catenary, viz., its equation is

$$\text{when}\ \exp.\theta=e^{\theta},\qquad y=\frac{c}{2}\left(\exp.\frac{x}{c}+\exp.\frac{-x}{c}\right)\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

If $(-\alpha, -\beta)$ be the coordinates of the vertex of the parabola, its equation is

$$(x+\alpha)^2 = 4a(y+\beta) \quad\text{.............................(2).}$$

Since the two curves have a contact of the second order, their osculating circles are the same at the point of contact. From (1), we have, for the catenary, $\dfrac{dy}{dx} = \dfrac{1}{2}\left(\exp.\dfrac{x}{c} - \exp.\dfrac{-x}{c}\right)$, $\dfrac{d^2y}{dx^2} = \dfrac{1}{2c}\left(\exp.\dfrac{x}{c} + \exp.\dfrac{-x}{c}\right)$,

$$1 + \left(\frac{dy}{dx}\right)^2 = \frac{1}{4}\left(\exp.\frac{x}{c} + \exp.\frac{-x}{c}\right)^2.$$

From (2), we have, for the parabola,

$$\frac{dy}{dx} = \frac{x+\alpha}{2a},\quad \frac{d^2y}{dx^2} = \frac{1}{2a},\quad 1 + \left(\frac{dy}{dx}\right)^2 = \frac{1}{4a^2}\left[4a^2 + (x+\alpha)^2\right].$$

Hence, from the usual formula for the radius of curvature, we see that, at the point of contact, the radii of curvature of the two curves are

$$\frac{c}{4}\left(\exp.\frac{x}{c} + \exp.\frac{-x}{c}\right)^2 \text{ and } \frac{1}{4a^2}\left[4a^2 + (x+\alpha)^2\right]^{\frac{3}{2}}$$

respectively, and, these values must be equal. Hence,

$$a^2c\left(\exp.\frac{x}{c} + \exp.\frac{-x}{c}\right)^2 = \left[4a^2 + (x+\alpha)^2\right]^{\frac{3}{2}} \quad\text{..................(3).}$$

Put $$\exp.\frac{x}{c} + \exp.\frac{-x}{c} = u,\quad x+\alpha = v \quad\text{..................(4, 5).}$$

Then, from (3), $$a^{\frac{4}{3}}c^{\frac{2}{3}}u^{\frac{4}{3}} = 4a^2 + v^2 \quad\text{................................(6),}$$

and, from (1) and (2), $$v^2 = 2a(cu + 2\beta) \quad\text{................................(7).}$$

Eliminating v^2, we get a quadratic for u, involving β alone, whence, solving, we have $u = F(\beta)$, say. Substituting in (7), we see that $v^2 = 2a\left[cF(\beta) + 2\beta\right]$, so that $v = f(\beta)$, suppose. But, from (4) and (5),

$$\exp.\frac{v-\alpha}{c} + \exp.\frac{-v+\alpha}{c} = u.$$

Hence, finally, the required equation of the locus of the vertex is

$$\exp.\frac{f(\beta)-\alpha}{c} + \exp.\frac{-f(\beta)+\alpha}{c} = F(\beta),$$

α, β being the current coordinates.

[Without bringing in the radius of curvature, the solution may be simply obtained from the fact that the values of y, $\dfrac{dy}{dx}$, $\dfrac{d^2y}{dx^2}$, corresponding to the same values of x, are the same in the two curves. Taking the usual equations

$$\frac{y}{c} = \frac{1}{2}\left(e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right)\quad (x-\alpha)^2 = 4a(y-\beta) \quad\text{.........(1, 2),}$$

we have $$\frac{dy}{dx} = \frac{x-\alpha}{2a} = \frac{(y^2-c^2)^{\frac{1}{2}}}{c},\quad \frac{d^2y}{dx^2} = \frac{1}{2a} = \frac{y}{c^2} \quad\text{..............(3, 4);}$$

from (1), (2), (3), (4) eliminate x, y, a, the resulting equation in α, β is that of the locus required. It readily follows that $y^2 - 2\beta y + c^2 = 0$, and thence the solution is easy, though the result, which is of the form given by the Solver, is complicated.]

8062. (By ASPARAGUS.)—The locus of the intersection of normals to a given conic drawn at the ends of a chord passing through a given point is in general a cubic. Is there any position of the given point (other than the centre of the given conic) for which the locus degenerates in degree?

Solution.

Let $$S = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1, \quad S' = 2\,(c^2xy + b^2y'x - a^2x'y)\,;$$

then, it is well known that the points on S at which the normals pass through the given point $x'y'$, are determined as the intersection of S with S'. Expressing the condition that the equation

$$\Delta S'^3 - \Theta S'^2 S + \Theta' S' S^2 - \Delta' S^3 = 0,$$

which represents the three pairs of lines joining the four points of intersection of the two conics, may be satisfied by the coordinates α, β of the given point, the locus is found to be

$$4a^4b^4\,(a^2\beta x - b^2\alpha y - c^2\alpha\beta)^3 + a^2b^2(a^2x^2 + b^2y^2 - c^4)\,(a^2\beta x - b^2\alpha y - c^2\alpha\beta)$$
$$(b^2\alpha^2 + a^2\beta^2 - a^2b^2)^2 + a^2b^2c^2xy(b^2\alpha^2 + a^2\beta^2 - a^2b^2)^3 = 0,$$

which is a cubic. This locus reduces in degree when $\alpha = \beta = 0$, or the given point is the centre; it reduces also in two other cases, viz., (1) the locus becomes a conic when the point is infinitely distant, that is, when we have to find the locus of the intersection of normals at the extremities of a chord which is parallel to a given line. (2) The locus becomes a conic when either $\alpha = 0$ or $\beta = 0$, that is, when the given point is on either axis; this, of course, includes the particular case when the given point is either the centre or either of the foci, which latter case may also be solved directly. All this, again, is a particular case of the more general property noticed by Mr. R. A. ROBERTS, that the locus of the intersection of lines making *any* constant angle (in this particular case $= \frac{1}{2}\pi$) with a conic at the extremities of a chord passing through a fixed point, is a cubic, the locus degenerating into a conic, not only when the fixed point is at infinity, but also when the fixed point is on the diameter which cuts the curve at the given angle, since the diameter is in this case part of the locus, as, indeed, is geometrically evident.

8103. (By ASPARAGUS.)—Given a system of confocal conics (foci S, S', centre C) and a point O, the well-known envelope of the polar of O is a certain parabola of which CO is directrix: prove that, if OL, OM be the tangents to this parabola from O, L, M will be the centres of curvature at O of the two conics of the system which pass through O.

Solution.

In my solution of Question 8129 [see p. 148 of this Volume], I have shown that the equation of the enveloping parabola is

$$(mx + ny)^2 - 2c^2\,(mx - ny) + c^4 = 0 \quad \text{.....................(1)},$$

whereof the parameter is $4c^2mn \,/\, (m^2 + n^2)^{\frac{3}{2}}$, the axis is the line $mx + ny = c^2\,\dfrac{m^2 - n^2}{m^2 + n^2}$, the directrix is $my = nx$, which is the line CO, and the focus is

the point $c^2\dfrac{m}{m^2+n^2}$, $-c^2\dfrac{n}{m^2+n^2}$. If we now take the two confocals through O, their equation is $\dfrac{x^2}{A^2+\lambda^2}+\dfrac{y^2}{B^2+\lambda^2}=1$ ....(2),

the double value of λ^2 being given by $\dfrac{m^2}{A^2+\lambda^2}+\dfrac{n^2}{B^2+\lambda^2}=1$ ....(3),

or, for shortness, writing $m^2=p\,(A^2+\lambda^2)$, $n^2=q\,(B^2+\lambda^2)$, this is $p+q=1$. Now, if ξ, ζ be the coordinates of the centre of curvature of (2) at O, we have $\xi=\dfrac{c^2m^3}{(A^2+\lambda^2)^2}$, $\zeta=\dfrac{-c^2n^3}{(B^2+\lambda^2)^2}$, or, $m\xi=c^2p^2$, $n\zeta=-c^2q^2$, and, in order that (ξ,ζ) may be on the parabola (1), we must have

$$T=(p^2-q^2)^2-2\,(p^2+q^2)+1=0,$$

when $p+q=1$. But, in fact,

$$\begin{aligned}T&=(p-q)^2-2\,[(p+q)^2-2pq]+1\\&=(p+q)^2-2\,(p+q)^2+1=-(p+q)^2=0\,;\end{aligned}$$

so that, the centres of curvature at O are on the parabola. Now, the tangents to the parabola at (ξ,ζ) are

$$m^2\xi x+n^2\zeta y+mn\,(\zeta x+\xi y)-c^2m\,(x+\xi)+c^2n\,(y+\xi)+c^4=0,$$

where, as before, $m\xi=c^2p^2$, $n\zeta=-c^2q^2$. If this tangent is to pass through (m,n), we must have the identical relation

$$m^2\,(p^2-q^2-1)+n^2\,(p^2-q^2+1)-c^2\,(p^2+q^2-1)=0,$$

or, since $p^2-q^2-1=-2q$, $p^2-q^2+1=2p$, $p^2+q^2-1=2pq$, $\dfrac{m^2}{p}-\dfrac{n^2}{q}=c$, which is identically true. Hence, we finally infer that the centres of curvature are on the parabola, and that the tangents to the parabola, at these points, pass through O.

8123. (By Professor LLOYD TANNER, M.A.)—Assuming the Moon to move round the Earth at a mean distance of 240,000 miles in 27 days 8 hours, and Jupiter's inner satellite to move round Jupiter at a mean distance of 260,000 miles in 1 day 18½ hours, compare the masses of Jupiter and the Earth.

Solution.

In my solution of Question 8044 (see Vol. XLIII., p. 112) I have completely proved the formula, $\log\mu=2\,(\log T-\log t)+3\,(\log r-\log R)$.

Here, $\mu=\dfrac{\text{mass of Jupiter}}{\text{mass of Earth}}$.

R = Earth's distance from Moon = $24\cdot10^4$ miles.
T = Moon's periodic time = 656 hours.
r = Distance of satellite from Jupiter – $26\cdot10^4$ miles.
t = Satellite's periodic time = 42·5 hours.

$$\log T = 2\cdot8169038 \qquad \log R = 5\cdot3802112$$
$$\log t = 1\cdot6283889 \qquad \log r = 5\cdot4149733,$$

whence $\log\mu = 2\cdot4813161$, which gives $\mu = 302\cdot9117482$.

This value of μ agrees fairly with the one given in LOCKYER'S *Astronomy*, p. 329, when $\mu = 300\cdot857$. The two results may be thus summarised, E. $=\cdot003301311$. J. E. $=\cdot003323916$. J.
These agree to the fourth decimal place, or the second significant figure.

8124. (By Professor COCHEZ.)—Trouver une courbe dont le rapport de son rayon de courbure à sa normale soit égal à 1 : μ.

Solution.

As regards the next question, we remark that the radius of curvature and the normal may lie on the same or on different sides of the curve; this will be indicated by taking the radius of curvature with a negative or a positive sign. Hence, by the condition,

$$y\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}} = \mp\mu\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}} + \frac{d^2y}{dx^2} \text{ or } \frac{d^2y}{dx^2} + 1+\left(\frac{dy}{dx}\right)^2 = \pm\frac{\mu}{y}.$$

Multiplying by $\dfrac{dy}{dx}$ and integrating,

$$\log\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}} = \mu(\log c \mp \log y), \text{ or } \left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}} = \left(\frac{c}{y}\right)^{\mu} \text{ or } (cy)^{\mu},$$

whence $$\frac{dy}{dx} = \frac{(c^{2\mu}-y^{2\mu})^{\frac{1}{2}}}{y^{\mu}}, \text{ or } (c^{2\mu}y^{2\mu}-1)^{\frac{1}{2}}.$$

To integrate the first, put $y = c^{2\mu}\cos^2\theta$, which transforms the integral

into $dx = -\dfrac{c}{\mu}(\cos\theta)^{\frac{1}{\mu}}\,d\theta$, whence $x+k = -\dfrac{c}{\mu}\displaystyle\int(\cos\theta)^{\frac{1}{\mu}}\,d\theta$,

which can be integrated by the ordinary formula of reduction, and can be finitely expressed whenever $1/\mu$ is a positive integer. Similarly, to obtain the second solution, we assume, $c^{2\mu}y^{2\mu} = \sec^2\phi$, which transforms

the equation into $$dx = \frac{1}{c\mu}(\cos\phi)^{-\frac{1}{\mu}}\,d\phi,$$

whence $$x+k = \frac{1}{c\mu}\int(\cos\phi)^{-\frac{1}{\mu}}\,d\phi.$$

A particular case of special interest is when $\mu = 1$, or the absolute value of the radius of curvature is equal to that of the normal. The first solution gives $(x+k)^2+y^2 = c^2$, which is a circle of radius c. The second solution

gives $\log\dfrac{y+(y^2-a^2)^{\frac{1}{2}}}{a} = \dfrac{x+k}{a}$, where $ca = 1$ and k is a new constant.

From this it follows that $y = \dfrac{a}{2}\left(e^{\frac{x+k}{a}} + e^{-\frac{x+k}{a}}\right)$, which is the equation to the catenary.

8127. (By Professor HADAMARD.)—Si A, B, C sont les angles d'un triangle, les angles λ, μ, ν, que font entre elles les médianes de ce triangle, sont donnés par les formules

$$\cot\lambda = \tfrac{1}{3}(\cot A - 2\cot B - 2\cot C), \qquad \cot\mu = \tfrac{1}{3}(\cot B - 2\cot C - 2\cot A),$$
$$\cot\nu = \tfrac{1}{3}(\cot C - 2\cot A - 2\cot B).$$

Solution.

We have $\angle BOC = \lambda$, $\angle OBC = \theta$, $\angle OCB = \phi$; then

$$\cot\lambda = \frac{1 - \cot\theta\cot\phi}{\cot\theta + \cot\phi} \quad \text{.........................(1)}.$$

Now, from elementary trigonometry, $\left(\frac{b}{2} + \frac{b}{2}\right)\cot B = \frac{b}{2}\cot\theta - \frac{b}{2}\cot C$,

whence $\cot\theta = \cot C + 2\cot B$. Similarly, $\cot\phi = \cot B + 2\cot C$,

Therefore $\cot\theta + \cot\phi = 3(\cot B + \cot C)$.

Also,
$$\begin{aligned} 1 - \cot\theta\cot\phi &= 1 - (2\cot^2 B + 2\cot^2 C + 5\cot B\cot C) \\ &= 1 - \cot B\cot C - 2(\cot B + \cot C)^2 \\ &= \cot A(\cot B + \cot C) - 2(\cot B + \cot C)^2. \end{aligned}$$

Substituting in (1), we have $3\cot\lambda = \cot A - 2\cot B - 2\cot C$, and similarly for $\cot\mu$, $\cot\nu$.

8129. (By Professor WOLSTENHOLME, M.A., Sc.D.)—Given a point O and a system of confocal conics (foci S, S′, centre C), if OP, OQ be tangents to any one of these conics, and through each point of PQ there be drawn a straight line perpendicular to its polar with respect to this conic; prove that (1) the envelope of all such straight lines is definite (the parabola which is also the envelope of PQ and of the normals at P and Q); (2) the locus of the point where each straight line meets its polar is also definite (being the circular cubic which is the locus of P, Q and of the foot of the perpendicular from O on PQ); (3) this locus and envelope depend only upon the relative positions of O, S, S′, although there are in each case two parameters involved, which we may take to be a/b, the ratio of the axes of the conic, and Y'/X' where $(X'Y')$ is the point on PQ through which the perpendicular is drawn.

Solution.

Let a conic of the system be $\quad \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ ....(1),

wherein, for convenience, I write $a^2 = A^2 + \lambda^2$, $b^2 = B^2 + \lambda^2$, where λ is the parameter of the system, and A, B the semi-axes of the primitive conic; so that $SS' = a^2 - b^2 = A^2 - B^2 = c^2$. Hence, if the given point O be (m, n), the equation of PQ which is the polar of O is

$$\frac{mx}{a^2} + \frac{ny}{b^2} = 1 \quad \text{...............................(2)}.$$

Now, if the coordinates of any point R on PQ be (α, β), the equation

$$\frac{m\alpha}{a^2} + \frac{n\beta}{b^2} = 1 \quad\text{.................................(3)}$$

is satisfied, and the polar of R (α, β), with regard to the conic, being

$$\frac{\alpha x}{a^2} + \frac{\beta y}{b^2} = 1 \quad\text{.................................(4),}$$

the line through R, at right angles to (4), is

$$\frac{\alpha}{a^2}(y-\beta) = \frac{\beta}{b^2}(x-\alpha) \quad\text{..............................(5)}$$

and we have to find the envelope of (5), when the variable parameters α and β satisfy (3). This may be done, as usual, by differentiating (3) and (5), and then employing LAGRANGE's *Method of Undetermined Multipliers*. As, however, the required envelope is of the second class, it is found with equal ease by getting rid of one of the variables, viz., eliminating β between (3) and (5), we have $\frac{c^2}{a^2}m\alpha^2 - \{mx + ny + c^2\}\,\alpha + a^2x = 0$, which gives for the envelope $(mx+ny+c^2)^2 = 4mc^2x$, which is equivalent to

$$(mx+ny)^2 - 2c^2(mx-ny) + c^4 = 0 \quad\text{.....................(6),}$$

a parabola of which the parameter is $\frac{4c^2mn}{(m^2+n^2)^{\frac{3}{2}}}$, the axis is the line $mx+ny = c^2\,\frac{m^2-n^2}{m^2+n^2}$, the directrix is $nx = my$, which represents CO, and the focus is the point $\left(c^2\,\frac{m}{m^2+n^2},\ -c^2\,\frac{n}{m^2+n^2}\right)$. If ξ, η be the coordinates of the focus, we see that $\xi^2+\eta^2 = \frac{c^4}{\delta^2}$, where $\delta^2 = m^2+n^2$; whence it appears that if the given point O describes circles concentric with the ellipse, the focus of the parabolic envelope also describes concentric circles. That this parabola is also the envelope of PQ for different members of the confocal family, is easily seen, viz., PQ being

$$\frac{mx}{A^2+\lambda^2} + \frac{ny}{B^2+\lambda^2} = 1,$$

the equation of which the envelope is to be found is

$$\lambda^4 + \{A^2+B^2-(mx+ny)\}\,\lambda^2 + A^2B^2 - B^2mx - A^2ny = 0,$$

which gives for the envelope

$$\{A^2+B^2-(mx+ny)\}^2 = 4\,\{A^2B^2 - B^2mx - A^2ny\},$$

which may be put into the form

$$(mx+ny)^2 - 2\,(A^2-B^2)\,(mx-ny) + (A^2-B^2)^2 = 0 \quad\text{...........(7),}$$

which, since $A^2-B^2 = a^2-b^2 = c^2$, represents the same parabola as in (6). Again, if the point P be (ϕ), the normal is

$$\frac{ax}{\cos\phi} - \frac{by}{\sin\phi} = c^2 \quad\text{.............................(8);}$$

and the tangent PO passing through O leads to the condition

$$\frac{m}{a}\cos\phi + \frac{n}{b}\sin\phi = 1 \text{ ....}(9).$$

Eliminating ϕ between (8) and (9), and putting $\lambda^2 = a^2 - A^2 = b^2 - B^2$, the envelope of (8) is found to be the same parabola.

Again, if we seek the locus of the intersection of the lines (4) and (5), which include a right angle, we have to eliminate α, β between (3), (4), (5). Now, from (3) and (4), we obtain the equations

$$\alpha = a^2 . \frac{n-y}{nx-my}, \quad \beta = b^2 . \frac{m-x}{my-nx}.$$

Hence, substituting these values for α, β in (5), we get for the required locus the cubic

$$(x^2+y^2)(nx-my) = (mx+ny)(nx-my) - c^2(m-x)(n-y) \text{......}(10).$$

Again, if we seek the foot of the perpendicular from O on PQ, we see that PQ is

$$\frac{mx}{A^2+\lambda^2} + \frac{ny}{B^2+\lambda^2} = 1 \text{ ....}(11);$$

and the line at right angles is

$$\frac{m}{A^2+\lambda^2}(y-n) = \frac{n}{B^2+\lambda^2}(x-m) \text{ ....}(12);$$

whence, solving for $A^2+\lambda^2$, $B^2+\lambda^2$, we have

$$A^2+\lambda^2 = \frac{m}{x-m}[x^2+y^2-(mx+ny)], \quad B^2+\lambda^2 = \frac{n}{y-n}[x^2+y^2-(mx+ny)],$$

which, by subtraction, give

$$A^2 - B^2 = c^2 = \frac{my-nx}{(x-m)(y-n)}[x^2+y^2-(mx+ny)],$$

which shows that this locus is the same cubic as (10). We can also find the locus of P in the same manner, namely, putting P as (ξ, η), we have the equations

$$\frac{\xi x}{A^2+\lambda^2} + \frac{\eta y}{B^2+\lambda^2} = 1, \quad \frac{m\xi}{A^2+\lambda^2} + \frac{n\eta}{B^2+\lambda^2} = 1 \text{...}(13, 14),$$

together with the conditions

$$\frac{x^2}{A^2+\lambda^2} + \frac{y^2}{B^2+\lambda^2} = 1, \quad \frac{\xi^2}{A^2+\lambda^2} + \frac{\eta^2}{B^2+\lambda^2} = 1 \text{.........}(15, 16),$$

from which ξ, η, λ are to be eliminated, viz., solving for ξ, η from (13), (14), we have

$$\xi = (A^2+\lambda^2)\frac{n-y}{nx-my}, \quad \eta = (B^2+\lambda^2)\frac{m-x}{nx-my},$$

which, being substituted in (16), gives for λ^2 the value

$$\lambda^2[(m-x)^2+(n-y)^2] = (my-nx)^2 - A^2(n-y)^2 - B^2(m-x)^2,$$

whence

$$A^2+\lambda^2 = \frac{(my-nx)^2+c^2(m-x)^2}{(m-x)^2+(n-y)^2}, \quad B^2+\lambda^2 = \frac{(my-nx)^2+c^2(n-y)^2}{(m-x)^2+(n-y)^2},$$

which, being substituted in (15), give for the required locus the same cubic as (10). It may be noted that the fact of the coincidence of the loci of the intersection of the polar of R with the perpendicular from R on it, and of

the intersection of the perpendicular from O on PQ with PQ, might have been *à priori* expected from the fundamental theorem in polars that, as R is on the polar of P, P is always on the polar of R, so that the properties are, in a sense, reciprocal. From a mere inspection of (6) and (10), it is evident that the envelope and the locus depend simply on the relative position of S, O, S′, since, wherever a, b occur, they occur in the form $a^2-b^2 = c^2 = OS^2 = OS'^2$. What makes the question peculiarly interesting is the determinateness, which we had *no* right to *expect* as *à priori* possible.

8144. (By Asparagus.)—Two points P, Q are taken on the coordinate axes conjugate to each other with respect to a conic U,

$$(a,\ b,\ c,\ f,\ g,\ h\between x,\ y,\ 1)^2 = 0;$$

prove that the envelope of PQ is the conic $(gx+fy+c)^2 = 4\,(fg-ch)\,xy$.

[This envelope is independent of a, b, which seems very singular. It degenerates when $ch = fg$, that is, when Ox, Oy are conjugate with respect to U; is an ellipse when $fg\,/\,ch > 1$, an hyperbola when $fg\,/\,ch < 1$.]

Solution.

Let O be the origin of the coordinate axes, and take $OP = m$, $OQ = n$, on these axes to which the conic

$$U = ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

is referred; then, the equation of PQ is

$$\frac{x}{m} + \frac{y}{n} = 1 \qquad\qquad (1).$$

Again, the polar of any point (x_1, y_1), with respect to $U = 0$, being

$$axx_1 + h\,(x_1y + xy_1) + byy_1 + g\,(x + x_1) + f\,(y + y_1) + c = 0,$$

either of the conditions

$$\left.\begin{matrix} x_1 = m, & x = 0 \\ y_1 = 0, & y = n \end{matrix}\right\}; \qquad \left.\begin{matrix} x_1 = 0, & x = m \\ y_1 = n, & y = 0 \end{matrix}\right\};$$

leads to the equation $\qquad hmn + gm + fn + c = 0 \qquad\qquad (2);$

and we have to find the envelope of (1) when the parameters m, n are connected by the relation (2). Eliminating n, we have

$$(hy+g)\,m^2 - (gx-fy-c)\,m - cx = 0,$$

the envelope of which is

$$(gx-fy-c)^2 + 4cx\,(hy+g) = 0, \quad \text{or} \quad (gx+fy+c)^2 = 4\,(fg-ch)\,xy.$$

It will be noticed that the absence of a, b from the envelope arises from the fact that *both* the given points are on the coordinate axes, which makes the coefficients of x^2 and y^2 vanish identically. The envelope degenerates into the polar of the origin when $fg = ch$, or, when the axes are conjugate to $U = 0$, is an ellipse or hyperbola, according as $fg > < ch$.

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